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FOUNDATIONS

*Beth, E. W. **Summulae Logicales. Supplement der Formele Logica. [Summulae Logicales. Supplement to Formal Logic].** Bibliotheek voor de Didactiek van de Exacte Vakken, no. 1. Noordhoff, Gronigen, 1942. 55 pp. (Dutch)

Diese Einführung (deren Titel einem Werke des Petrus Hispanus, 1226–1277, entlehnt ist) in die formal logische Methode ist für Nicht-Mathematiker und demgemäss sehr ausführlich geschrieben. Für den Kenner des Gebietes sind vor allem die zahlreichen historischen Bemerkungen interessant. Der Verfasser behandelt den Propositionenkalkül und den Prädikatenkalkül ausführlich und ganz kurz auch die Relationslogik. Die traditionelle Syllogistik wird in ihren Beziehungen zur modernen Methode recht gründlich untersucht. *H. Freudenthal* (Amsterdam).

*Feys, R. **Logistiek. Geformaliseerde Logica. I. Algemeen Overzicht. Propositie- en Klassenlogica. [Logistic. Formal Logic. I. General Survey. Logic of Propositions and Classes].** Philosophische Bibliotheek, N.V. Standaard-Boekhandel, Antwerp, 1944. 340 pp. (Dutch)

Es handelt sich um den ersten Band eines ausführlichen Handbuchs der Logistik (ohne Anwendungen, auch nicht auf die Mathematik). Auf die zahlreichen originellen Bemerkungen und Ansätze kann in Rahmen dieser Besprechung nicht eingegangen werden, doch wollen wir darauf hinweisen, dass Interessenten für die Geschichte der formalen Logik hier allerlei Material finden können. Der Verfasser skizziert erst die Entwicklung der formalisierten Logik bis zur modernen Modalitätslogik. Er gibt eine sehr ausführliche Übersetzung der Schlussformen der klassischen Logik in die moderne logische Sprache. Nach der historischen Einleitung geht er über zur Beschreibung des zu entwickelnden logischen Systems, wobei er systematisch die metalogischen Begriffsbildungen und Methoden auseinandersetzt. Er geht dabei auf die Formalisation der Metatheorie mit all ihren Problemen ein, jedoch nicht auf die eigentliche Metamathematik.

Auf diesen immer noch einleitenden Teil folgt der systematische: erst der Satz- und dann der Klassenkalkül. Der Verfasser beginnt mit dem Aufbau auf Grund von Wahrheitsfunktionen, geht dann axiomatisch vor, und er entwickelt auch beide Methoden von Gentzen für den Satzkalkül. Vom Klassenkalkül wird in diesem ersten Teil noch die Logik der Funktionen von Individual-Variablen behandelt. *H. Freudenthal* (Amsterdam).

Quine, W. V. On ordered pairs. J. Symbolic Logic **10**, 95–96 (1945).

Previous methods of defining ordered couples have made an ordered couple always at least one type higher than its components. In the present note the author shows how an ordered couple can be defined so as to be of the same type as its components. This construction can be carried out in practically any system of set-theory in which there are no individuals. *J. C. C. McKinsey* (Reno, Nev.).

Barzin, Marcel. Logique symbolique. Sur la portée du théorème de M. Gödel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **26**, 230–239 (1940). [MF 13832]

The author tries to show that a pair of formulas given by Gödel [Monatsh. Math. Phys. **38**, 173–198 (1931)] leads to a contradiction. In the reviewer's opinion, the author has fallen into error because of his attempt to simplify Gödel's notation. In particular, to take a crucial point of the argument, it seems most unlikely that the second formula on page 236 could be derived from the first if both were transformed back into Gödel's original notation. [The second of these formulas, indeed, appears to be meaningless, since it involves quantification with respect to "σ," which is supposed to be the Gödel-number of the variable "s," and hence itself a constant.] *J. C. C. McKinsey* (Reno, Nev.).

Rosser, J. B., and Turquette, A. R. Axiom schemes for m-valued propositional calculi. J. Symbolic Logic **10**, 61–82 (1945).

This paper is concerned with the axiomatization of systems of sentential calculus which are defined by matrices, that is, with the problem of finding finite sets of axiom schemes from which just those formulas are deducible by means of modus ponens which always give designated values. The authors first give a set of axiom schemes for the Łukasiewicz-Tarski sentential calculus. The proof that this set of axiom schemes is adequate is made obscure by the fact that, in the definition of an S-sum at the bottom of page 69, a variable (α) occurs in the definiens which does not occur in the definiendum; this defect could most likely be corrected, however, even if at the expense of lengthening the proof. [Although the authors give credit to Wajsberg for having given an axiomatization of this calculus for the case $m=3$, it should be pointed out that Lindenbaum and Wajsberg have also proved that the Łukasiewicz-Tarski system is axiomatizable for a general m [see J. Łukasiewicz and A. Tarski, C. R. Soc. Sci. Varsovie, Cl. III **23**, 30–50 (1930); M. Wajsberg, Monatsh. Math. Phys. **42**, 221–242 (1935)].]

The authors then extend this result to systems based on two sentential connectives C and N (as in the Łukasiewicz-Tarski system) but with more than one designated value. Finally, they give sets of axiom schemes for more general systems (involving more than two sentential connectives). *J. C. C. McKinsey* (Reno, Nev.).

Destouches-Février, Paulette. Rapports entre le calcul des problèmes et le calcul des propositions. C. R. Acad. Sci. Paris **220**, 484–486 (1945). [MF 14044]

The author describes a method which, she claims, will generate a calculus of problems from any given calculus of propositions. It is then stated that: (1) the strongest possible calculus of problems is that defined by Kolmogoroff; (2) the strongest calculus of propositions isomorphic to its associated calculus of problems is the Heyting calculus,

and in this case the associated calculus of problems is that of Kolmogoroff; (3) every calculus of propositions weaker than the Heyting calculus is isomorphic to its associated calculus of problems.

These results can hardly be said to have been proved in the present paper, or even to have been formulated in a clear way, since the notion of an associated calculus of problems is not made precise. If "p" is a sentence, for example, it is not clear whether "Pb(p)" is supposed to be a sentence or an entity of some other sort. From the text one would judge that "Pb(p)" is not intended to be a sentence, and otherwise it would seem more reasonable to use the equivalence sign, instead of the identity sign, in the first part of axiom 4b; but, on the other hand, it appears that axiom 5 cannot make sense unless "Pb(p)" and "Pb(q)" are sentences. *J. C. C. McKinsey* (Reno, Nev.).

de Bengy-Puyvallée, Renaud. Sur les règles de composabilité dans les logiques de la complémentarité de M^me^ Destouches-Février. C. R. Acad. Sci. Paris **220**, 589–591 (1945). [MF 14058]

Calling experimental propositions $p_1, \cdots, p_n$ "composable" when they assign values to magnitudes which quantum theory allows to be simultaneously measurable, the author indicates this property by writing $\neq(p_1, \cdots, p_n)$ and asserts some postulates regarding it. The letters "p," "q," "r" occurring in these postulates are apparently permitted to take as values only propositions recording immediate experimental observations, and thus not, for example, propositions involving negation, disjunction, etc. It is not clear what kind of properties are to be assumed here for the ordinary sentential constants "$\sim$", "$\rightarrow$", etc. That these are not necessarily the usual ones is shown by the author's remark following his second postulate: "This postulate is superfluous if one has $\sim\sim r \cdot \equiv \cdot r$ as a rule for negation." This uncertainty makes it difficult to comprehend or evaluate the postulates regarding "$\neq$", since these involve also the ordinary sentential constants. [It is perhaps worth pointing out that, if one supposes that an experimental proposition can be false, and if one assumes that a false proposition implies every proposition and that the conjunction of two true propositions is true, then from postulate 7, by replacing "p" by a false experimental proposition, one can show that every two experimental propositions are composable.] *J. C. C. McKinsey* (Reno, Nev.).

McKinsey, J. C. C. On the syntactical construction of systems of modal logic. J. Symbolic Logic **10**, 83–94 (1945).

The author studies the properties of modal logics in which the notion of possibility is defined syntactically. The basic idea is to consider a sentence possible if it has the same form as some true sentence, that is, if it becomes true after substituting the proper constants for some of the nonlogical symbols in it. This makes the notion of possibility depend to some extent on the substitutions that are available. The author constructs in this way a type of modal logic called a system T_2 and shows that all provable formulas of the Lewis system of strict implication S4 are provable in a system T_2, but that not all provable formulas of the Lewis system S5 are provable in T_2. He also investigates the special case of a system which has symbols for the integers and for the relation of equality between integers. If a certain formula F which is provable in this system is added as a postulate to the postulates for S4, a system S4.1 results which is distinct from all the Lewis systems, and is neither stronger nor weaker than S5. It is shown that there are ten distinct modalities in this new system S4.1.

O. Frink (State College, Pa.).

Gokieli, L. On the separation of quantities into classes with the help of reflexive, symmetric and transitive relations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 493–502 (1944). (Georgian and Russian) [MF 14599]

Gokieli, L. On the present idea of the infinitely small. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 11–20 (1944). (Georgian and Russian) [MF 14598]

Gokieli, L. P. On the notion of existence in mathematics. III, IV. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **13**, 153–206 (1944). (Russian. Georgian summary)

Parts I and II appeared in Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 881–888 (1941); **3**, 111–118 (1942); these Rev. **5**, 198.

Cartan, Henri. Sur le fondement logique des mathématiques. Revue Sci. (Rev. Rose Illus.) **81**, 3–11 (1943). [MF 13808]

This is a popular exposition of the logical foundations of mathematics. The author is concerned primarily with the Zermelo axiomatic set theory. He contrasts its exact methods with the vague views of Borel, Hadamard and Lebesgue concerning definition and mathematical existence. He then outlines the Zermelo theory in essentially the form due to Fraenkel, where the underlying logic is presumably intended to be the functional calculus of first order. [The author is not careful to distinguish between mention and use of expressions. In the discussion of existence and uniqueness there is occasional omission of essential quantifiers; the discussion would have been facilitated by explicit reference to Russell's theory of descriptions or to Hilbert's ι-function. The statement of the Skolem paradox is somewhat misleading and does not sufficiently distinguish this paradox from that of Richard.] *R. M. Martin* (Chicago, Ill.).

Sebastião e Silva, José. On the axiomatic method. Gaz. Mat., Lisboa **6**, no. 26, 2–3 (1945). (Portuguese) [MF 15115]

Expository article.

Williams, Donald. On the derivation of probabilities from frequencies. Philos. and Phenomenol. Res. **5**, 449–484 (1945). (English. Spanish summary) [MF 13384]

This and the eight papers reviewed below form a symposium on probability. Inasmuch as the papers are largely of the nature of a debate, it has been expedient to make each review contain some material from the others: a complete picture of each requires the reading of all.

Williams states that his interest in probability has two philosophical purposes: to show how nondemonstrative knowledge can be rationally cogent, and to promulgate a world view which shall be cogent in a higher degree than any of its rivals (and a world view, he says, is validable, if at all, only with an inductive probability). The collective (or frequency) theory he regards as "lying athwart the one corridor which leads to any philosophy which is not sheer animal habit, and to any way of life which is more than

fanaticism or blind convention." Exactly what is the true analysis of probability, and whether and in what sense it does validate inductive inference, are questions which, he states, he will postpone. It may be remarked at the outset that most of the subsequent authors in this symposium are not in essential disagreement with this objective, although they have in general a higher regard for collectives.

Williams asserts in this paper, and reaffirms in his later one, that he understands the "classical" meaning of probability to coincide with the more or less popularly accepted and intuitively understood idea summoned up by such expressions as "degree of rational belief," "likelihood," "reasonable expectancy," etc., in short, "probability" in the sense of the man in the street. This coincides with Carnap's "probability$_1$" and the reviewer's "intuitive probability" [with the mathematical theory of which Williams seems to be unfamiliar]. And he holds that this is the sense in which inductions (nondemonstrative inferences) have to be made, and that it is essentially a logical notion. Without this concept there is no induction and hence no science. We must return to this conception and thereby "return the discussion of probability to the proper philosophical arena." But Williams also uses the expression "classical theory of probability" to mean the Laplacean theory, with its ratio of favorable to possible cases. Sometimes it appears that he regards the Laplacean theory as furnishing the appropriate mathematical formulation of the more general logical notion; at other times he seems to be noncommittal on this point and to introduce Laplace as a tu quoque in arguments against collectivism. Indeed a large part of the paper can be interpreted as either disparaging collectivism because it is as bad as the worst that has been said about Laplaceanism or extolling the latter because it has all the virtues of collectivism without its vices.

To give one example among many of such arguments, Williams says that collectivism makes as much use of the principle of indifference as Laplace, if it is to be taken as applying to an individual case. And if the collectivists avoid the difficulty (as most of them do) by asserting that their theory, while the only scientific theory of probability, does not apply to an individual case, we are confronted with the absurdity that there is nothing in the scientific theory of probability which can make me as an individual decide whether to insure my life or not, or which has any relation to testing any particular theory of nature.

Collectivists have objected to the "classical" theory as being less in touch with experimental reality than their own. Williams' answer: Firstly, if being "more real" means dealing with an aggregate of substantival objects (occurrences or things, actual or potential) rather than adjectival ones (qualities), the distinction is on the face of it unconvincing to common sense and not validable logically (logical mechanisms allow the conversion of substantival and adjectival into one another). Secondly, the infinite collective is the apotheosis of idealization. [Nagel will later retort that many an essential physical concept is an idealization involving infinite sequences and limits, in which tu quoque he has apparently missed Williams' point, which was a defense of the "reality" of classical probability by a tu quoque.] In this connection, Williams remarks that, if collectivism is viewed as of like kidney to "operational" science and an essential part thereof, it automatically becomes its very Achilles' heel, "since on the same principle by which it excludes any valid induction to future frequencies, it excludes induction to any of the trains of future experiences which constitute all physical facts for the avowed operationalist."

Williams places great emphasis on the two well-known and in the opinion of many (including the reviewer) fatal objections to pure collectivism: from a finite segment, however long, of a collective, nothing can be inferred as to the value of the limiting frequency (except in terms of a noncollective kind of probability); and from a postulated value of limiting frequency, nothing can be inferred (with the above exception) concerning any segment open to human observation.

In his defense of the Laplacean theory, Williams answers the objection to the usually required "equiprobability" of the possible cases, and seems to vindicate the principle of indifference, by asserting that the theory applies whenever the only "relevant" knowledge is that there are n mutually exclusive alternatives, r of which are favorable. Now in every application of the theory there is further knowledge, and the decisive need of saying whether it is or is not "relevant" is in the minds of many (including the reviewer) the fatal objection in principle to the theory regarded as an embodiment of the concept of classical (logical) probability, as distinguished from affording an occasionally available method of measurement of its numerical value. This objection will be pointed out by Nagel, Reichenbach, and others. They will also object to Williams' dismissal of the Bertrand paradox (in the Laplacean treatment of a continuum of alternatives). Williams is satisfied in this paper with saying that the paradox is involved just as much by collectives (which contention is shown to be wrong by Reichenbach); and in his subsequent paper, he feels that such probabilities are really not used in practice. [This is surprising to a reviewer who has witnessed the almost daily application of probability to such questions as antisubmarine warfare, in which geometrical probabilities (in principle, of a nonfrequency type) were indispensable.]

Apart from the objections of the last two paragraphs, Williams' points appear to the reviewer to be both valid and important. In spite of the fact that all of them have been known to science for decades at least, Williams is right in repeating them, since so many collectivists and others seem to be ignorant thereof. *B. O. Koopman.*

Nagel, Ernest. Probability and non-demonstrative inference. Philos. and Phenomenol. Res. **5**, 485–507 (1945). (English. Spanish summary) [MF 13385]

In this second paper of the symposium on probability, Nagel undertakes to defend the collective or frequency theory from Williams' attack; then, to make comments in outline on a second conception of probability of a noncollectivist type appearing in arguments which, while they may impart a degree of conviction, are not and cannot be made logical proofs: "non-demonstrative arguments"; finally, to point out the [in our mind fatal] objections to the Laplacean theory regarded as the rendering of the fundamental concept of probability. Nagel enters a strong objection to Williams' determination to regard "probability" as having only one true meaning: he declares himself at least a dualist (and possibly a pluralist) in this regard, and believes that at least two meanings must be recognized as valid: the collectivist meaning and that of non-demonstrative inference. This dualism will be emphasized in a far clearer light in Carnap's paper.

Nagel shows that the numerical values of some probabilities are in practice obtained as frequency ratios, whence

the notion of collective is obtained by the type of idealization common in mathematical physics. This is intended to refute Williams' thesis "that the collective theory is a mistaken description of the exact internal composition of that property ('credibility') which probability denotes—whether used by frequency theorists or by ordinary people." Nagel's example certainly indicates that frequency furnishes an idealized physical constant, naturally equated to the numerical value of a probability. But in itself it does not prove that this collectivist conception is identifiable with the epistemological concept of probability [a point clearly brought out by Carnap, and noted by Williams in his later paper]. Yet it would appear to be in the latter sense that Williams' thesis is uttered; indeed he explicitly acknowledges the possible utility of frequencies in measurement. The reader will have to decide whether Nagel has indeed refuted Williams or has merely disagreed with his terminology.

To the familiar objection that between the limiting frequency in the collective and the finite frequency in its humanly accessible segment no strict logical relation exists, Nagel answers by analogy (following von Mises and others): other physical quantities (density, velocity, etc.) are limits of quotients, only a finite set of values of which are accessible to measurement; are we to reject them for that reason? The only logical answer is "yes, if a strict logical connection between observations and postulated idealizations of the physical world is required." This is the very answer Nagel seeks to elicit, since it compels the admission that the argument of an absence of strict logical relationship between observed quotients and their limits does not invalidate frequency any more than density or velocity. But there is a fundamental implication which the pure frequentists have never grasped, but which Nagel with his pluralistic use of "probability" seems to be on the verge of admitting, without ever being quite unequivocal on the matter: if the conceptions of density and velocity as limiting quotients are useful in physics in spite of the aforementioned lack of strict logical relation between them and the measurements, it is because it is felt that they have nevertheless a relationship of "probable" numerical agreement with one another, "probable" in an intuitive sense, similar to Nagel's "nondemonstrative inference." Thus Nagel's argument by analogy, if it is to salvage frequency, has the consequence that it makes intuitive probability a presupposed element of thought in any application of collectives to experimental reality. This would appear to be related to Williams' thesis quoted in the preceding paragraph. This situation will be pointed out clearly by Carnap.

Nagel declares that it is a mistake to suppose that probability statements construed in the frequency sense can be tested only on the basis of statistical evidence. Thus a certain type of electronic change of probability (limiting frequency) p might be connected with a quantity c (for example, current strength) measurable directly on an individual (not as a limiting frequency). If general physical theory shows that c varies in a definite way with p (sic), measurement of c (an individual operation) provides data for computing p. Evidently the author has in mind some physical theory other than quantum mechanics or classical statistical mechanics, within which such a situation is impossible in principle. Current strength in quantum mechanics is just another quantum number; its mean, like p, can be measured only statistically, although its dispersion is much smaller. Of course p can be calculated from the wave equation; but it is a question here of experimental testing.

Nagel's final defense of frequency by relating it to the "formalistic" introduction of probability by Lebesgue measure-like postulates (Kolmogoroff) reposes on the questionable assumption that a property whose probability is close to 1 will occur.

In Nagel's discussion of the second meaning which he admits for probability (non-demonstrative inference, etc.) he passes various conceptions of this kind in review (including Keynes's, Jeffreys', and the reviewer's [who objects, however, to the statement that his theory requires some form of the principle of indifference]). The Spielraum theory and its difficulties and Peirce's suggested theory of testing are briefly considered. À propos of this noncollectivist conception of probability, Nagel raises the questions: is such a quantity capable of numerical measurement? of explicit definition? etc., but apparently without realizing that the authors he cites had gone pretty far toward giving answers to all these questions, although often from radically different points of view from one another. *B. O. Koopman.*

Reichenbach, Hans. Reply to Donald C. Williams' criticism of the frequency theory of probability. Philos. and Phenomenol. Res. **5**, 508–512 (1945). [MF 13386]

Reichenbach asserts that he thinks that none of Williams' criticisms of the frequency theory can stand the test of logical analysis, but that he considers that the detailed refutation of Williams is more or less contained in his various writings on the subject. He confines himself to a brief outline of refutation of what he terms the core of the usual criticisms of the frequency interpretation of probability, adding: "I cannot say to what extent they are shared by Williams because I do not always understand his exposition."

On the negative side, Reichenbach refutes the use of the principle of indifference, by recalling the contradictions to which it gives rise, both in the case of finitely many alternatives and in the case of a continuum of alternatives (geometrical probability and Bertrand's paradox). He asserts that "the cases in which it [the principle of indifference] leads to a true conclusion turn out, in closer analysis, to be of such a kind that much more is known to us than is stated in Laplace's assumption, and the fallacious inference, therefore, constitutes a fallacy of incorrect schematization; . . .," with all of which the reviewer agrees.

On the positive side, in defense of the author's interpretation of collectivism, he says that he regards the proposition that only frequencies in collectives can furnish a directive for future action as the fundamental thesis of the frequency theory, "since preferring the more probable event (i.e., of higher frequency) means having success in the greater number of cases." But the frequency of an event in an infinite collective is not the measure of the "number" of occurrences of the event in the collective, since by an easily defined rearrangement of the elements of the collective it is possible in all nontrivial cases to change the value of the frequency. Frequency and "numerosity" correspond only in finite collectives. It would thus seem that from the author's own point of view the distinction between finite and infinite collectives is the very reverse of being "only a matter of mathematical convenience," as he terms it. Later, Reichenbach admits that the inductive inference from the frequency in the observed part of the collective (whether infinite, or finite and large) is the most serious difficulty of the theory; but he considers that it has been overcome by his theory of induction "which shows that the inductive inference, conceived as a method of posit and

later correction, must lead to success if success is attainable" (reference to his Wahrscheinlichkeitslehre), adding: "I have seen no serious objection to this theory of induction." If this were a review of the cited work, the reviewer would undertake to fill this lacuna by showing that the author's own interpretation of his own claims is not given a complete mathematical proof. *B. O. Koopman.*

Carnap, Rudolf. The two concepts of probability. Philos. and Phenomenol. Res. **5**, 513–532 (1945). (English. Spanish summary) [MF 13387]

In arguments about the foundations of the subject, as they occur, for example, in most of the papers of this symposium, the issue seems to be "how to define probability." This may appear bizarre, inasmuch as "probability" is only a word and its use is purely a matter of convenience, unworthy of heated disagreement. Carnap throws a clear light on the situation: the issue is not in the definition of a blank symbol (word), but in finding an appropriate precise and explicit conception (the "explicatum") to use in place of a vague but vivid one current in popular or pre-scientific discourse (the "explicandum"). Thus, thermometrically defined "temperature" is the explicatum appropriately replacing or complementing sensually perceived "hotness and coldness." Carnap observes that, while the suggested explicata for probability are many and more or less successful, the explicanda are essentially always two in number.

The first explicandum (probability$_1$ or "degree of confirmation") corresponds to the more or less vague and intuitive idea of likelihood, degree of rational belief, probability in its common usage. This, as Carnap is very thorough in showing, belongs to logic. It is a relation between factual statements (propositions or sentences), a relation which is irrelevant to the actual or supposed truth or falsity of the latter, just as the relation of implication applies or does not apply according to the meaning of the propositions and without regard to their factual correctness. He distinguishes three orders of this degree of confirmation: (1) "hypothesis h is supported by evidence e"; (2) "h is more strongly confirmed by e than h' by e'"; (3) "h is confirmed to the extent p (a number between 0 and 1) by e." [Cf. the reviewer's notation: $h/e>h/\sim e$ for (1), $h/e>h'/e'$ for (2), and $p(h/e)$ for (3), in his publications cited below.]

The second explicandum (probability$_2$ or "relative frequency in the long run") corresponds to the notion of richness of an outcome of a stated sort in a multitude of trials under stated conditions. It is a physical parameter, and an assertion that it has a certain value is an assertion of physical fact.

Carnap shows by a thorough discussion of typical contentions of the various schools of thought that the chief source of disagreement is the more or less implicit tendency to recognize one of these two explicanda and not the other. Yet they are both indispensable to science. The references to Carnap given in the last three reviews make it unnecessary to go into more detail here.

Carnap points out the weaknesses of the explicatum proposed by Williams and others for probability$_1$, but announces that he will shortly publish a satisfactory one. Carnap refers to collective theory as an adequate explicatum for probability$_2$, provided that it operates in conjunction with probability$_1$, rather than trying to eliminate it.

The author discusses the "psychologism" claimed by many as a basis of probability$_1$ as well as of deductive logic and says that, if one examines the methods of those making such claims, it becomes apparent that they are really objectivist and do not actually make psychological experiments. But he does not say whether the view that "knowledge of the laws of logic is gained by awareness of one's own mental processes" implies a psychological experiment or an objectivist method.

Apart from the latter obscurity, the reviewer agrees with Carnap in toto. [Cf. B. O. Koopman, Bull. Amer. Math. Soc. **46**, 763–774 (1940); these Rev. **2**, 106 for a general statement and references to papers giving detailed treatment and establishing the reciprocal relationship between probability$_1$ and probability$_2$.] *B. O. Koopman.*

Margenau, Henry. On the frequency theory of probability. Philos. and Phenomenol. Res. **6**, 11–25 (1945). (English. Spanish summary) [MF 14272]

Nagel and Carnap have emphasized the duality of the concept of probability. Margenau likewise stresses this duality but, in distinction to the former authors, his duality is not between the logical conception (Nagel's nondemonstrative inference, Carnap's probability$_1$, the reviewer's intuitive probability, etc., all corresponding to the classical "degree of rational belief") on the one hand, and the idealized-experimental conception of frequency or collectives on the other: Margenau's duality is between probability according to a "constitutive" definition (typified by Laplace's "proportion of favorable cases") and according to an "operational" definition (of which "frequency in a collective" is a primary example). Explicitly avoiding philosophical (and hence, apparently, logistical) definitions, the author proposes to confine himself to scientific ones, namely, to those which the scientist may be seen actually to use in his work, as distinguished, perhaps, from his philosophical reflections about science.

Margenau asserts that the quantities which occur in science all require a constitutive definition ("mass is the symbol m which appears in Newton's second law of motion") and an operational definition ("mass is measured in such and such a manner with balances"). The latter establish the correspondence with experiment; the former, with the (mathematical) theories. Only when both definitions are at hand can science set its intricate machinery in motion and perform its amazing feats. The author hastens to emphasize that physical quantities can be expected to have many constitutive definitions (as many as there are theories in which it plays a major role) and many operational definitions (depending on convenience, convention, and manipulative skill). And their number and form will change with the development of science: a sign of the fruitfulness of the interplay of the two fundamental forms of scientific definition. He also emphasizes that this interplay is so intricate that it is often difficult to state whether a given definition is of one type or the other. It may, indeed, be wondered whether Margenau has in fact established a dichotomy between the definitions, rather between aspects of one and the same definition. The reviewer knows of no operational definition which does not repose in an essential way upon a theory and hence have an unavoidably constitutive side, and vice versa. Indeed, this difficulty could be brought out by a detailed critique of the very example of the volume of the sphere, developed by the author to clarify his distinction.

The crucial application of all the foregoing is to Margenau's identification (and justification, within the limits of its applicability) of Laplace's definition with a constitutive definition of probability, and of frequency as the opera-

tional definition. Yet again there is the difficulty that when the former can be applied, the total number of cases and of favorable cases can be counted, which makes Laplace's definition operational; whereas the theory of frequency in sequences is as much a mathematical theory as Euclidean geometry, thus making collectivism's definition a constitutive one (this point is brought out in von Mises' paper, reviewed below). Thus the author's introductory statement, that the conflict between the collectivists and their opponents is but a minor phase of the battle between the empiricist and the rationalist, seems to require more clarification than he has given it.

The reader will have to judge whether Margenau has indeed fulfilled his own program: to define what the working scientist of probability actually uses pertaining to this concept. Does the scientist (or the practical man, for that matter) not use the "probability of a single event"? Is such a use invalid and to be discarded, as Margenau says, as the price of securing scientific precision for the theory? And does the scientist not reason inductively and thus employ probability as a branch of logic? And would not the recognition of such a conception clarify those difficulties with operational definitions in general, and frequencies in particular, which the author either fails to mention, or does so only to leave in obscurity? These questions, having been treated at length in earlier papers of this symposium (and in their reviews), are not pursued further here. Some of the author's statements concerning quantum physics will be objected to by many physicists.

B. O. Koopman (New York, N. Y.).

Bergmann, Gustav. Frequencies, probabilities, and positivism. Philos. and Phenomenol. Res. 6, 26–44 (1945). (English. Spanish summary) [MF 14273]

Bergmann states that probability theory in general and frequency theory in particular belong to the philosophy of science, that particular branch of philosophy which furnishes an analysis "of the concepts and methods of science as used by science from a basis of common sense realism." He adds: "This notion of the philosophy of science, if taken seriously, has two consequences. First, its analyses are logically or, if you please, metaphysically quite unproblematic and non-controversial. Conversely, they do not prejudge one's philosophical views." Example: "Einstein's analysis of nonlocal simultaneity . . . has not, as is sometimes claimed, 'proved' any form of philosophical empiricism, subjectivism, or positivism; it has, in particular, not disproved ontological realism." Second, "a philosophical position that cannot or will not 'square itself' with the results of the philosophy of science does not meet one of the requirements any philosophical position must meet if it is to deserve serious consideration." These two consequences are brought out by his elaboration of the example of local simultaneity. The bearing of all this on the present controversy concerns Williams' objection to the frequency theory as claiming to be more scientific and operational and less subjective than Laplace's theory. For, on the one hand, frequency theory properly understood (and Bergman shares Williams' objection to many of the "philosophical" claims of certain frequentists, at least in their form) could not in the nature of things have any such implication; and, on the other hand, it is valid to ask how the Laplacean can know from his (perfectly scientific and operational) knowledge that, if a quarter of the marbles in this bag are red, then, if a great number of drawings (with replacement, etc.) are made, red will appear about a quarter of the time. Williams' attack on frequency "because he wishes to undermine the philosophical subjectivism, experimentalism, operationism, positivism, or phenomenalism . . . which he believes to be the philosophical presupposition of the frequency theory" is quite beside the point. The philosophical phase of Williams' argument is regarded by Bergmann as an example of the confusion due to the "mixing of levels of analysis." Any reader to whom all this is not clear is advised to read Bergmann's paper, as well as his many other papers upon the knowledge of which he depends heavily.

Returning to the bag of marbles question, Bergmann attempts to envision just what Williams thinks can be deduced from the bag's composition. Williams had declared that we can at most predict what the results of a series of drawings "will probably be." But is this a probability in the sense of Laplace? No, says Bergmann, who seems unacquainted with the interpretation of Bernoulli's theorem within the framework of Laplace's definition. And he declares that the conception of "will probably be" as an undefinable has been excluded by Williams, as well as the analysis of Wittgenstein-Waismann. He feels that Williams gives little clarification of what the expression really does mean.

The author gives a solution of difficulties ("more apparent than real") which seem to be involved in applying frequency theory to nonordered mass events (involving great numbers of cases), by showing how a collective of such events can be formed, and then replaced by a supercollective, that is, collective of fictional collectives. All this has a bearing on Williams' criticism of frequentists because of their arbitrary restricting of their "reference classes." Williams has a point here, he says, since frequentists are "not always as explicit as philosophers could desire."

Coming to Williams' main criticism of frequency theory, that from the finite initial segment open to observation nothing can be learned concerning frequency limits in the collective, except by an "induction" giving only the values which they "probably" have (in some nonfrequency sense of "probably"), Bergmann seems on the point of giving a refutation. But he really never does: he seems to miss the point of the criticism entirely and to be led by the word "induction" into a general epistemological discussion of "induction" in an entirely different sense of the term: the induction of scientific theories from experimental data. To be sure, this is relevant to a different anti-frequency argument of Williams, that it provides no method of measuring the probability of such inductions, "since universes are not as plentiful as blackberries." In his discussion, Bergmann depends at so many essential points upon his earlier works (without giving their content) that we must omit further details.

Bergmann believes that the mathematical difficulties of collective theory have been dealt with very effectively and hence are not, as Williams holds, under "adjudication." This view of Bergmann's is encountered in many of the other papers of this symposium, the authors often referring to the results of Copeland and Wald as removing all mathematical objections. The question at issue is that of defining the class S of selection rules (mathematically formulated rules for picking a subcollective from a collective). Originally, von Mises had included in S so many rules that no collective could exist. This led his successors to a restriction of S, but without at first being able to show that the corresponding collectives exist even then. The proof that, under

liberal definitions of S, they do exist was Copeland and Wald's contribution. But the difficulty remains that any so restricted S can always be increased by adding equally natural selection rules, and without invalidating the Copeland-Wald conclusion; hence any given S is an artificially restricted S, and the whole collectivist attempt to define the randomness of a sequence mathematically involves this apparently ineradicable artificiality. But the appreciation of this question involves more mathematical considerations than many of the contributors to the symposium seem willing to give.

Bergmann says in conclusion: "No matter what probability theory one holds, the statement about our bag full of marbles, or about the drawings from it, contains factual information. So it speaks either about such things as beliefs, or about the external world, or, possibly, about both. If it is taken as a statement about beliefs, then its correct translation lies in the pragmatic metalanguages. There is undoubtedly such a 'subjective' aspect to many statements people make in which either the word 'probable' or one of its colloquial equivalents occur. But I consider it as quite obvious and noncontroversial that some such statements contain factual information about the external world. With this meaning or connotation of the term the frequency theory is concerned. Obviously, this meaning cannot be expressed by one of the characteristic predicates of the pragmatic languages, for these do not speak about the external world. Nor can 'probable' be either a syntactical or a semantical predicate, for syntax and semantics do not contain any factual information whatsoever. This precludes assimilation of 'probable' to 'reasonably believable,' 'true,' and 'analytic.' The theories of Ramsey, Keynes, and Wittgenstein-Waismann are thus excluded. So the meaning of 'probable' for which we are looking must lie in the object language." *B. O. Koopman* (New York, N. Y.).

v. Mises, R. Comments on Donald Williams' paper. Philos. and Phenomenol. Res. 6, 45–46 (1945). [MF 14274]

The author repeats the well-known objections to the Laplace theory of probability: the Bertrand paradox in the continuous case, and the objection, in the finite one, that "Williams simply skips its essential part, namely, the condition that the favorable and all possible cases must be equally likely cases. If this is added, the definition appears as a vicious circle, since 'equally likely' means having the same probability. If it is omitted, the definition becomes completely senseless, since nobody would contend that the probability of casting five is $\frac{1}{6}$ or the probability of an odd number $\frac{1}{3}$ when gambling with an arbitrarily shaped body of six distinct faces." And he adds that Laplace's definition cannot handle the case of a slightly loaded die.

In defending his collective theory of probability against Williams' accusation of not being "empiristical" or "operational" as it pretends to be, but using Platonic ideas fatal to such qualities, von Mises states that "the collective theory claims to be a mathematical theory of mass phenomena of the same character as geometry is the theory of spatial phenomena and mechanics the theory of motion of visible bodies. It is exactly to the same extent empirical (and nonempirical) as any of these or other branches of theoretical physics. . . ."

Neither Williams nor any other contributor to the symposium has expressed any objection to this account of collective theory [certainly the reviewer has not]. The objection is simply to von Mises' exclusion of any other conception of probability from the ranks of scientific theories ("scientific" is used here in the broad sense and includes mathematics and logic, not in the narrow sense of exclusively laboratory science); and to von Mises' lack of appreciation of the fact that such an exclusion would completely sever the mathematical theory of collectives from observational phenomena, just as it would divide the observed from the idealized quantities in any physical theory. von Mises says that he has "stressed over and over again that the collective theory covers only a part of the field of phenomena in which the term 'probability' is used in colloquial language." However, it seems to the reviewer that whenever von Mises has admitted a noncollective meaning of probability it is always to relegate it to the vague folk-language denoting a concept which could not in the nature of things form the proper subject of a scientific theory.

The issue of chief significance is between the probabilistic monism of such collectivists as von Mises, Reichenbach, etc., and of such classicists as Keynes and Jeffreys, on the one hand, and the dualism of Nagel, Carnap, and Koopman, on the other. *B. O. Koopman* (New York, N. Y.).

Kaufmann, Felix. Scientific procedure and probability. Philos. and Phenomenol. Res. 6, 47–66 (1945). [MF 14275]

After a first part devoted to a review of some of the important features of the seven previous papers of the symposium, Kaufmann devotes the rest of his paper to the logic of inductive inference. He attaches this to Carnap's discussion of probability$_1$ (degree of confirmation), but takes issue with Carnap at many points. In particular, he rejects the notion of invariable truth of a synthetic proposition, and regards the statement "k gives (inductive) confirmation to h" as being an abbreviation of the statement "it is correct in terms of the presupposed rules of induction R to add h to S, if k belongs to S," where R is a set of rules and S a set of admitted synthetic propositions. He holds that induction is very different from deduction, in that in the latter an implication is derived by an examination of meanings, whereas an inductive confirmation goes outside of these and presupposes a body of knowledge and of rules. "In contrast to deductive inference, it [inductive inference] does not reveal an internal relation between the propositions connected by the rules. We derive the proposition: 'h may be added to S' from the rule: 'If k belongs to S then h may be added to S' and the statement 'k belongs to S.'" But since Kaufmann never gives even a single concrete example of such R and S, the usefulness of his point of view is not clear.

Much of the paper involves metaphysical theses concerning truth and knowledge and, like so many metaphysical discussions, does not prove, but rather affirms, a particular position. The feature of the paper most relevant to probability is the author's rejection of the idea that induction establishes a (varying degree of) probability of a conclusion. Instead, he claims that an induction decides us to admit the conclusion provisionally into the body of accepted conclusions; that later evidence (that is, augmentation of this body) may induce us subsequently to reject it. Thus Kaufmann falls into the trap set for all those who attempt to evade probability by describing some of the actions of scientists who are drawing tentative conclusions, while averting his glance from other (but relevant) phases of their mental processes, their essential reason for working on the basis of the tentative conclusion, namely, that they regard it as "probable." But he is scarcely unequivocal even in this

regard, since at the end of the paper he says: "In distinguishing incompletely confirmed statements from completely confirmed (i.e., warranted) statements we might employ the word 'likelihood' and the corresponding adjective 'likely,' . . ."

Other items: Kaufmann rejects Reichenbach's "identity conception of probability," without more than the vaguest indication of his reasons. And he states [in diametrical opposition to the facts] that the reviewer's theory may be interpreted as an elaboration or modification of the classical (Laplacean) doctrine. *B. O. Koopman.*

Williams, Donald. The challenging situation in the philosophy of probability. Philos. and Phenomenol. Res. **6**, 67–86 (1945). [MF 14276]

At the outset of this concluding paper to the second part of the symposium on probability, Williams declares that "all sane men, at any rate, acknowledge explicitly or implicitly that some propositions in given circumstances are better supported, evidenced, warranted, or confirmed, more reasonable, more logical, more tenable, than others: that they possess, in short, higher degrees of probability. The philosophical theory of probability is the part of logic and epistemology which attempts to analyze this delicate and indispensable principle. This is what I meant when I wished that the problem be returned 'to the proper philosophical arena,' . . ." He holds that only confusion results from the use of the term "probability" in any other sense. And he asserts that "the 'classic' or 'Laplacean' theory, the 'frequency theory,' and the other theories of probability which I mentioned, are rival accounts of the nature of graded credibility." He then reformulates the Laplacean thesis and passes in review the eighteen points made in his first paper, in part I of the symposium, points in which the principal contentions of the advocates of a frequency theory received comment, usually unfavorable; this was to show that "as a complexion theory of probability [that is, one involving the notion of proportion of objects, events, or outcomes, in a finite or infinite aggregate] the frequency theory is distinctly less promising than the more general classic theory. . . ."

While holding this view concerning the inadequacy of the frequency theory as an account of the one meaning which he allows for "probability," Williams is at great pains to assert (in both his papers) that he has no quarrel with the collectivist theory of frequencies and that the phenomena of frequency play an essential role in the experimental sciences: it has never been a "frequency theory of frequencies" to which he has objected (except that he has "less than Mr. Nagel's and Mr. Bergman's confidence that Wald has provided a definitive rule for rulelessness," for which he has the reviewer's blessing), but a "frequency theory of probability." Carnap's distinction is thus emphasized; yet "Mr. Carnap is still too lenient by half when, proposing 'probability$_1$' for probability, he would admit 'probability$_2$' for frequency"; and: "Mr. Carnap is too charitable again when he writes as if each party understood quite clearly their own end (to investigate probabilities or frequencies, as the case might be), and go wrong only by ascribing to the other party the same end, with competing means."

Williams observes (correctly, in the reviewer's opinion) that many of the criticisms directed against his first paper can be classified as follows: (1) irrelevant: defending only the frequency theory of frequencies, which he never attacked; (2) erroneous: attacking statements which he never made; (3) intimidative: consisting of such statements as that Williams is just in a general way wrong, that there has been no serious objection to the "well established" position of certain frequentists, that their position is supported by "well-known" but otherwise unspecified arguments which Williams has not considered, and finally that Williams' attitude "smacks of the nineteenth century"; (4) iterative: merely repeating without apology many of the same arguments whose refutation they were called upon to refute. This whole situation is developed in detail with rich and specific illustration.

It is in his affirmation of the Laplacean conception that Williams appears to the reviewer to expose himself to the greatest possibility of criticism. The set P being a subset of the set M (for example, the total possible exclusive outcomes of a trial), he states in general: "Given that m/n M is P, and that a is [in] M (and given nothing else that is relevant), 'the' probability that a is [in] P equals m/n." [The brackets are the reviewer's.] It is precisely the statement in parentheses which is the weakness of the definition. For in every actual case there is more information about a than what is given above, and the fact that it can so often be regarded as "irrelevant" is what makes possible the many fruitful applications of this definition, regarded as a method of numerical appraisal. Yet the decision to regard a piece of information as "irrelevant" is in substance to regard its presence as not changing the probability of the proposition "a is in P," that is, to assert that this probability is not only meaningful but is, given the information, equal to what it would be, not given the information. Thus the Laplacean construction, while a useful method of deriving probability statements from other (simpler) probability statements, does not allow the derivation of probability statements from data fundamentally devoid of all probability statements. [This, according to the reviewer's position, would be impossible in principle.] Later, concerning von Mises' statement that "nobody would contend that the probability of casting five is 1/6 . . . when gambling with an arbitrarily shaped body of six distinct sides," Williams declares this is exactly what he would do, given only that the body will turn up one of six sides and that there is a five on just one of them. "Only when there is given not merely that the body is irregular, but how it is irregular, and how that kind of irregularity affects its behavior, is the probability different from 1/6."

The possibility of such further increment to our information must not prevent us from ascribing the 1/6 probability before such information has been vouchsafed, any more than the truism that probability changes with change in the body of knowledge concerning the event should lead to the abandonment of the whole of probability. The "principle of indifference" Williams regards as fundamentally nothing more than the above position. Later he says: "The common formula for 'non-sufficient reason,' that 'the classic theory assumes that the cases are equally likely unless they are known to be otherwise,' has a derogatory flavor, but it is literally true and harmless, when divorced from the absurd requirement that 'equally likely' mean equally frequent in all reference classes." The difficulty here, as the reviewer sees it, is that without any increment whatsoever to the body of knowledge (the "supposal") with reference to which the probability of a proposition (for example, "a is in P," in the earlier example) is being sought, it is always possible to form a different reference class (for example, to define a

new set M' of total mutually exclusive outcomes of our event by compounding some of the original outcomes in M with outcomes of a different independent and "neutral" event—of which set M', P is still a subclass, but where m'/n ($\neq m/n$) M' is P), and so to obtain a different probability. This is the essence of the "life on Mars" paradox, whose only resolution has been based on the assertion that when M is replaced by M', knowledge comes to hand which is "relevant" to upsetting the m'/n probability; but here again is the troublesome notion of the "relevant." If probability is to be regarded as relative not only to the supposal, but to the particular manner of analysis of our knowledge into disjunctions, then it will indeed be hard to see how it can ever be applied to actual cases, without "conventionalism" at its worst. *B. O. Koopman* (New York, N. Y.).

Bouligand, Georges. Sur quelques groupements de problèmes. Revue Sci. (Rev. Rose Illus.) **82**, 3–14 (1944). [MF 13800]

Après un rappel de généralités sur les problèmes, pris à titre individuel, plusieurs principes de groupement seront passés successivement en revue et illustrés par des exemples: 1° aptitude à la résolution par un type opératoire commun, recherchée dans un but de classement; 2° recours à l'idée de séparabilité et à quelques idées connexes en théorie des classes abstraites; 3° recours à l'idée de synopsis, dont s'inspirent les tables numériques, les procédés nomographiques et les groupements de questions réalisés sous une forme telle qu'il devienne indifférent de traiter l'une d'elles ou le problème inverse. *Extract from the paper.*

Hempel, C. G. On the nature of mathematical truth. Amer. Math. Monthly **52**, 543–556 (1945). [MF 14485]

Rossier, Paul. La science de l'espace. Arch. Sci. Phys. Nat., Geneva **25**, 91–105 (1943). [MF 14213]

***Henderson, Archibald. Mathematics and the physical sciences.** A State University Surveys the Humanities, edited by Loren C. MacKinney, Nicholson B. Adams, and Harry K. Russell, pp. 144–159. University of North Carolina Press, Chapel Hill, N. C., 1945. $4.00.

This is chapter 12 of part II, which is subtitled "The Humanities and the Humanistic Ideal in the Fields of University Education."

Destouches-Février, Paulette. Une nouvelle preuve du caractère essentiel de l'indéterminisme quantique. C. R. Acad. Sci. Paris **220**, 553–555 (1945). [MF 14055]

Destouches-Février, Paulette. Sur l'impossibilité d'un retour au déterminisme en microphysique. C. R. Acad. Sci. Paris **220**, 587–589 (1945). [MF 14057]

Beck, Guido. Mathematical formalism and the physical picture. Philos. Sci. **12**, 174–178 (1945). [MF 13932]

ANALYSIS

Calculus

Diaz, J. Gallego. On the permutation of the operators d/dx and E_x. Gaz. Mat., Lisboa **6**, no. 26, 1 (1945). (Spanish) [MF 15114]

The operator E_x carries y into xy'/y, where $y'=dy/dx$. The requirement that (*) $[E_x(y)]'=E_x(y')$ leads to the form $y=\exp\{\int\psi(e^x/x)dx\}$. The relation (*) is said to arise in mathematical economics. *R. P. Boas, Jr.*

Badell, Enrique. Contribution to the theoretical study of the reflection and refraction of light by means of electromagnetic theory. Revista Soc. Cubana Ci. Fís. Mat. **2**, 9–11 (1945). (Spanish) [MF 14181]

The problem mentioned in the title involves the use of the fact that the relation $Ae^{ax}+Be^{bx}+Ce^{cx}\equiv 0$ implies $A=B=C=0$. The author's proof of this well-known fact could have been abbreviated by using the properties of Vandermonde determinants. *R. P. Boas, Jr.*

***Trost, Ernst. Eine anschauliche Herleitung der Stirlingschen Formel.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 138–140, Füssli, Zürich, 1945.

The proof combines a geometric approximation to the area under the curve $y=\log x$ with estimates obtained from the power series for $\log\{(1+x)/(1-x)\}$. [For an even more geometrical discussion, see P. Puig Adam, Revista Mat. Hisp.-Amer. (3) **1**, 21–26 (1939); these Rev. **1**, 300.] *R. P. Boas, Jr.* (Providence, R. I.).

Hamel, Georg. Über die Umkehrung einer Potenzreihe. Deutsche Math. **5**, 338–339 (1940). [MF 14338]

An elementary derivation of an upper limit for the radius of convergence of the Maclaurin expansion for the inverse of an analytic function $z=f(w)$, when $f(0)=0$, $f'(0)\neq 0$. *P. Franklin* (Cambridge, Mass.).

Hartman, Philip. Remark on Taylor's formula. Bull. Amer. Math. Soc. **51**, 731–732 (1945). [MF 13611]

The author shows that Taylor's formula

$$f(a)=\sum_{k=0}^{n-1}f^{(k)}(0)a^k/k!+f^{(n)}(\xi)a^n/n!,\qquad 0<\xi<a$$

holds whenever (i) $f(x)$ is continuous on the closed interval $[0, a]$, (ii) $f(x)$ has $n-1$ derivatives on the half closed interval $[0, a)$ and (iii) $f(x)$ has an nth derivative on the open interval $(0, a)$, thus demonstrating that the usual additional assumption that $f^{(n-1)}(x)$ is continuous at $x=0$ is extraneous. *W. T. Reid* (Evanston, Ill.).

Neville, E. H. Indefinite integration by means of residues. Math. Student **13**, 16–25 (1945). [MF 14490]

The value of $\int_0^\infty f(x^2)dx$ is determined by contour integration as the sum of the real parts of the residues of the function $f(z^2)\log\{(c+z)/(c-z)\}$ at those poles of $f(z^2)$ whose real parts are positive. The restrictions are made that $f(z^2)$ has no pole on the imaginary axis and that the integral of $f(z^2)\log\{(c+z)/(c-z)\}$ around a suitable semicircular contour tends to zero as the radius of the contour tends to infinity. The first restriction is removed by a suitable adjustment in the residue sum. The value of $\int_0^\infty f(x)dx$ is given as the negative of the sum of the residues of the function $f(z)\log(c-z)$ at all the poles of $f(z)$. For this result there is no requirement that $f(z)$ is even. The only restriction is that a certain contour integral tends to zero as the contour becomes infinite. Examples are given to indicate the manner in which the position and type of the individual poles affects the residue sum. *P. Civin.*

van Veen, S. C. Über vollständige elliptische Integrale, die durch Γ-Funktionen ausgedruckt werden. Mathematica, Zutphen. B. 12, 69–75 (1943). [MF 14305]

Proof that certain complete elliptic integrals of the first kind can be expressed in terms of beta functions.

P. Franklin (Cambridge, Mass.).

Thielemans, L. Sur l'évaluation de certaines intégrales définies. C. R. Acad. Sci. Paris 220, 422–424 (1945). [MF 13954]

The author considers the integrals

$$I_n=\int_L f(a_1t, a_2t, \cdots, a_nt)t^{-n}dt,$$

where n is a positive integer, the factors a_r are real and the path L is independent of the a_r. It is assumed that differentiation under the integral sign with respect to each a_r is legitimate. A reduction formula is obtained by integration by parts, use being made of the operator

$$(a\nabla)f=a_1\partial f/\partial(a_1t)+a_2\partial f/\partial(a_2t)+\cdots.$$

As an example, the integral

$$\int_0^\infty e^{-xt}\sin(\gamma t)\,\mathrm{sh}\,(at)\sin(bt)t^{-3}dt, \qquad x\geqq a>0,$$

is evaluated with the aid of logarithms and inverse tangents. [The article contains some misprints.]

H. Bateman (Pasadena, Calif.).

Humbert, Pierre. Sur une formule de M. Nielsen. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 61–63 (1940). [MF 13774]

For the Fresnel integrals $C(x)$, $S(x)$ Nielsen [Theorie des Integrallogarithmus und Verwandter Transzendenten, Teubner, Leipzig, 1906] has given the relation

$$C^2(x)+S^2(x)=\int_0^{x/2}e^{-x^2\tan\theta}\sin\tfrac{1}{2}\theta\,(\cos\theta)^{\frac{1}{2}}\csc 2\theta d\theta.$$

From the well-known formula

$$C^2(x)+S^2(x)=\tfrac{1}{4}\pi\,\mathrm{erfc}\,(x\sqrt{i})\,\mathrm{erf}\,(x\sqrt{-i})$$

the author infers that C^2+S^2 is not an even function of x and concludes erroneously that Nielsen's formula must be incorrect. By operational methods he gives an alternative integral representation,

$$C^2(x)+S^2(x)=\int_0^\infty e^{-2xu}\sin u^2u^{-1}du.$$

A. Erdélyi (Edinburgh).

Chern, Shiing-Shen. On Grassmann and differential rings and their relations to the theory of multiple integrals. Sankhyā 7, 2–8 (1945). [MF 13784]

This paper is expository in character and presents the essentials of a theory previously available only in isolated pieces. First, Grassmann rings are defined axiomatically. Then a realization of a Grassmann ring by means of determinants shows the axioms to be consistent. Next, differential rings are developed out of Grassmann rings. Finally, the theory of multiple integration is developed from this point of view. Particular reference is made to the transformation formulae of multiple integrals under change of coordinates and to Stokes's theorem. *C. B. Allendoerfer.*

van Veen, S. C. Riemann double integrals. Mathematica, Zutphen. B. 10, 40–52, 100–119; 11, 1–20, 27–34 (1942). (Dutch) [MF 14312]

Continuation of an article of which the first part appeared in Mathematica, Zutphen. B. 10, 1–17 (1941); these Rev. 3, 294.

Klose, A. Massfunktionen in der Vektorrechnung. Deutsche Math. 5, 322–328 (1940). [MF 14341]

The paper is a new form of presentation of the vector and tensor calculus, in which spherical averages of functions of unit vectors play a substantial role.

I. Opatowski (Chicago, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Olmsted, John M. H. Transfinite rationals. Bull. Amer. Math. Soc. 51, 776–780 (1945). [MF 13619]

Using a simple equivalence relation between ordered pairs of cardinal numbers, the author obtains a lattice R having operations of multiplication, division and addition, and such that (1) both the cardinal numbers and the positive rational numbers can be isomorphically imbedded in R with respect to addition, multiplication and order, (2) certain of the ordinary algebraic laws are valid in R and (3) R is the smallest system having these properties. *L. H. Loomis.*

Eyraud, Henri. La représentation des nombres ordinaux. C. R. Acad. Sci. Paris 218, 635–636 (1944). [MF 13472]

The author indicates an order-preserving method of representing the ordinal numbers of the second class by functions having range and domain in the positive integers.

L. H. Loomis (Cambridge, Mass.).

Bellon, Waldemar. Cantor, the conqueror of the infinite. Univ. Nac. Colombia 3, 353–373 (1945). (Spanish) [MF 13976]

This article is mainly an exposition of topics in the theory of sets and cardinal numbers.

Denjoy, Arnaud. Figuration des nombres transfinis de la classe II. C. R. Acad. Sci. Paris 221, 429–432 (1945). [MF 14683]

This communication concerns the representation by means of real numbers of the permutations of N: 1, 2, 3, $\cdots$ (that is, the rearrangements of N as linear orders). A permutation P of N is defined if, for every n, assuming P rearranges $1, 2, \cdots n-1$ as $u_1, u_2, \cdots u_{n-1}$, we know where n is placed relatively to the u_r (whether before u_1, or between u_1 and u_2, or $\cdots$ or after u_{n-1}). Every real number x of the segment $e-1\leqq x\leqq e$, where e is the base of natural logarithms, is developable in the form $\sum_1^n a_n/n!$, with $1\leqq a_n\leqq n$. These n possibilities for a_n match the n possible ways of placing n with respect to the u_r. The author adopts the following prescription for placing n when x, and hence $\{a_n\}$, is given: if $a_n\leqq n-1$, n comes immediately before the particular u_r which equals a_n; and if $a_n=n$, n comes after u_{n-1}. He then derives a necessary and sufficient condition that P, thus defined by means of $\{a_n\}$, is (1) a normal order, and (2) of order type ω. The results are allied to those of earlier papers of the author based on other prescriptions for placing n [cf. the same C. R. 212, 885–888 (1941); 213, 430–433 (1941); these Rev. 3, 73; 5, 113]. *H. Blumberg.*

Denjoy, Arnaud. Sur les ensembles cartésiens. C. R. Acad. Sci. Paris 221, 679–682 (1945). [MF 15155]

This note is concerned with the possibility of mapping one Euclidean space E_r onto another E_p in such a way that a given nonnegative additive set function in E_r corresponds

to Lebesgue measure in E_p. The mappings are not required to be one-to-one, but are supposed to preserve at least some topological properties. Various conditions are suggested which may require $p=r$. No proofs are given.

J. C. Oxtoby (Bryn Mawr, Pa.).

Maximoff, Isaiah. On the transfinite spaces E and on the continuum hypothesis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 231–233 (1944). [MF 12668]

In the space E_α^r of transfinite sequences of transfinite numbers the author previously [Ann. of Math. (2) **41**, 321–327 (1940); these Rev. **1**, 206] established an order relation which he uses in this paper to define closed, perfect, and hypermeasurable sets of E_α^r and continuous and hypermeasurable functions with domain and range both in E_α^r. These notions are then carried over to the space $E_\alpha^{r^k}$ whose points are transfinite sequences of points of E_α^r. The main result (although stated differently) is essentially that, if a function with domain and range in $E_\alpha^{r^k}$ is either continuous or hypermeasurable, then it does not carry a straight line through the origin into the set each point of which has only one coordinate different from the rational point $\{1\}$ of E_α^r and that coordinate the rational point $\{2\}$ of E_α^r. The contrapositive proof depends on continuum information from the previously cited paper and consists in showing that a certain set has power both less and greater than $2^{\aleph_0}$.

J. F. Randolph (Oberlin, Ohio).

Kurepa, Georges. À propos d'une généralisation de la notion d'ensembles bien ordonnés. Acta Math. **75**, 139–150 (1943). [MF 13207]

This article belongs to the series of the author's papers concerning the ramified tableaux and connected with Souslin's problem [in particular, Publ. Math. Univ. Belgrade **4**, 1–138 (1935); **6–7**, 124–160 (1937)]. A partially ordered set T is called a ramified tableau if for every point a of T the set of all points preceding a is well ordered. Besides three other conditions, let T satisfy one of the following conditions. (a) Every well ordered subset of T is at most denumerable. (b) For every $a \in T$, the set of all points of T comparable with a contains an ordered infinite nondenumerable set. (c) Every maximal ordered subset of T has a power at least $\aleph_1$. Then T is called a sequence (s), a sequence (S) or a sequence (C), respectively. The author gives some theorems and lemmas about such sequences. Two of them show that certain statements are equivalent to the affirmative answer of Souslin's problem or to the hypothesis of the continuum. Moreover, the author proves the existence of a minimal sequence (S) and of two sequences (S) which are different with respect to their order. The existence of such sequences (s) remains an open question. *A. Rosenthal.*

Cuesta, N. Continua of Souslin. Revista Mat. Hisp.-Amer. (4) **4**, 175–187, 215–233 (1944). (Spanish) [MF 12172]

The study of the problem of Souslin for the ramified tables of Kurepa [Publ. Math. Univ. Belgrade **4**, 1–138 (1935)] led the author to a theory of transfinite dyadics which are the subject of the present paper. A "number" of index α (α any finite or transfinite ordinal) is a function which to every $\beta<\alpha$ associates one of the two values 0 or 1. Every number A_α of index α defines a number A_β of any index $\beta<\alpha$, called a section of A_α. Any collection of numbers which with each number contains also all its sections is called a tree. The height of a tree is the least ordinal larger than all the indices of the numbers in the tree. The numbers in each tree can be ordered in a natural fashion. The author shows that every ordered set satisfying the conditions of the problem of Souslin is similar to a tree. This permits him to prove several properties of Souslin continua; for instance, the cardinal number of any one of them is $2^{\aleph_0}$. A problem is formulated whose negative solution would imply a positive solution of the Souslin problem.

S. Eilenberg (Ann Arbor, Mich.).

Besicovitch, A. S. Totally heterogeneous continua. Proc. Cambridge Philos. Soc. **41**, 96–103 (1945). [MF 12840]

A continuum is "totally heterogeneous" if there are no two points of it such that a neighborhood of one is homeomorphic to a part or the whole of a neighborhood of the other. The purpose of the paper is to prove the existence of totally heterogeneous continua. One step in the proof is the (negative) solution of the problem of Zarankiewicz [Fund. Math. **7**, 381 (1925)]. Given a dendrite D (that is, a locally connected continuum having no Jordan curve as subset), does there necessarily exist a subdendrite D_1 homeomorphic to D? The method employed for this solution is then developed to yield the principal result. [The problem of Zarankiewicz was solved earlier by E. W. Miller [Bull. Amer. Math. Soc. **38**, 831–834 (1932)].] *H. Blumberg.*

Keldych, Ludmila. Sur la structure des ensembles mesurables B. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 71–98 (1944). (French. Russian summary) [MF 12282]

This article is a summary of the author's researches over a lengthy period [preliminary notices have appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 523–525 (1940); **28**, 675–677 (1940); **31**, 651–653 (1941); these Rev. **2**, 256; **3**, 226; cf. also Z. P. Dienes, J. London Math. Soc. **14**, 169–175 (1939); these Rev. **1**, 8].

Following Lusin, attention is confined to the set I_α of all irrationals between 0 and 1 and an α-set is defined (for any $\alpha<\omega_1$) as the limit of a convergent sequence of β-sets ($\beta<\alpha$) (a 0-set is one which, together with its complement, is a sum of intervals). An α-element is a set which can be represented as a countable product of β-sets with $\beta<\alpha$. An α-element which cannot be represented as the countable sum of β-elements with $\beta<\alpha$ is said to be an α-element precisely.

There is little difficulty in determining the structure of α-elements in the cases $\alpha=0, 1, 2$. An elegant arithmetic example of a 3-element was given by Baire in 1905 but little progress was made in this theory until the author established, in 1934, that any 3-element is the sum of a countable system of α-sets ($\alpha=0, 1, 2$) and a 3-element homeomorphic to Baire's set. This set can therefore justly be regarded as a basic or canonical 3-element. The problem of extending this result can be solved completely.

An α-element E can be represented in terms of an A-system of sets $\{E(n_1, \cdots, n_k)\}$ in the form

$$E=\prod_k \sum_{(n_1, \cdots, n_k)} E(n_1, \cdots, n_k),$$

where $E(n_1, \cdots, n_{k+1}) \subset E(n_1, \cdots, n_k)$ and where each $E(n_1, \cdots, n_k)$ is a β-element ($\beta<\alpha-1$ if $\alpha-1$ exists, $\beta<\alpha$ otherwise). It is useful to consider E as given by a sieve whose components are sets of points (x, y) where $x \in E(n_1, \cdots, n_k)$ and where y belongs to a certain rational interval. This variation of the usual definition is convenient in the discussions of the operations on A-systems in chapter I.

The whole descriptive theory of Borel sets can be relativized with respect to an α-element E considered as generated by a definite A-system by making correspond to an interval $\delta(n_1, \cdots, n_k)$ the set $E \cdot E(n_1, \cdots, n_k)$. Conditions are obtained which are necessary and sufficient in order that a subset of an α-element should be an $(\alpha+1)$-element in terms of relative G_δ's and F_σ's.

In chapter I certain transfinite operations on A-systems are introduced; these correspond to displacements of components of the sieves. For each α there is defined a sequence of operations which will produce from the A-system defining an element E an A-system defining any set which is an α-element relative to E. The main result is that any α-element E can be regarded as being defined by means of a sieve constructed from the sieve which corresponds to any $F_{\sigma\delta}$ containing E.

Chapter II deals with the structure of α-elements. An α-element is called universal if, corresponding to any β-element e ($\beta \leqq \alpha$), there exists a perfect set P such that $P \cdot E$ is homeomorphic to e. An α-element is called canonical if it is, on each portion of its closure, universal and of the first category (an alternative definition, in terms of the corresponding A-system, is also given). The construction of canonical α-elements for $\alpha < \omega_1$ by induction from the case $\alpha = 3$ is sketched.

It is shown that two canonical α-elements are homeomorphic and that canonical elements are homogeneous in the sense that every portion (containing more than one point) is homeomorphic to the whole. It is also shown that canonical α-elements are α-elements precisely. The fundamental theorem is then established: any α-element ($\alpha \geqq 2$) can be represented as the sum of a canonical α-element E_α and an at most countable sum of α-elements none of which have points in common with E_α.

Two problems which remain open are: (i) if E is an α-element precisely can a canonical α-element be obtained by removing from E a sum $\sum e_\beta$ of β-elements ($\beta < \alpha$) such that $\sum e_\beta$ is an α-element? (ii) if E is an α-element precisely is it universal? An affirmative answer to (ii) implies an affirmative answer to (i); an affirmative answer to (i) will enable an intrinsic definition of an α-element precisely to be given.

Chapter III deals with arithmetical examples of canonical elements. The construction of α-elements and canonical α-elements given in chapters I and II, respectively, is effective provided that α is given effectively, that is, if the set of all numbers $\beta < \alpha$ is enumerated. Using Baire's example as a basis it is possible to characterize arithmetically the points of a canonical α-element. This is done in some detail in the cases $\alpha \leqq \omega$. *J. Todd* (London).

Albuquerque, J. Ensembles de Borel. Portugaliae Math. 4, 161–198 (1944). [MF 12952]

The Borel construction of repeated operations of countable sums of mutually exclusive sets and intersections of a set and the complement of a subset is carried through clearly and in detail starting with a fundamental family I_0 which is finitely multiplicative, contains the whole space as an element, and is not a σ-field. Assuming in addition that there is a monotone increasing function f with domain I_0 and range the extended nonnegative real number system, the technique of Borel is followed to define a measure function on Borel sets. [The assumption $f(S) < \infty$ for S the whole space seems to be necessary.] Outer and inner measures of other sets are defined by means of supersets and subsets. Measurability is defined and theorems given, without proof, which are equivalent to stating that measurability in the sense of Radon and in the sense of Carathéodory are equivalent. *J. F. Randolph* (Oberlin, Ohio).

Albuquerque, J. Ensembles de Borel. II. Portugaliae Math. 4, 217–224 (1945). [MF 13330]

Continuing the discussion of the paper reviewed above, the author makes an identification between the families of sets formed by the Borel method and the families classified by the method of Baire and de la Vallée Poussin.

J. F. Randolph (Oberlin, Ohio).

Neves Real, L. General theory of measure, 3. Borel measure. Introduction. Class B. Cadernos de Análise Geral, no. 14. Junta de Investigação Matemática, Pôrto, 1945. iv+35 pp. (Portuguese) [MF 13643]

Neves Real, L. General theory of measure, 4. Borel measure. Definition and general theorems. Cadernos de Análise Geral, no. 15. Junta de Investigação Matemática, Pôrto, 1945. 28 pp. [paged 35–62] (Portuguese) [MF 13644]

Neves Real, L. General theory of measure, 5. Lebesgue measure and Carathéodory measure. Cadernos de Análise Geral, no. 16. Junta de Investigação Matemática, Pôrto, 1945. 43 pp. [paged 64–106] (Portuguese) [MF 13645]

These three booklets contain a connected account of the theories of and relationships between Borel, Lebesgue, and Carathéodory measure and measurability. They are designed to follow two others of a series on the theory of measure, one a general introduction to the concepts of measure, the other a discussion of Jordan measure. The first of these booklets contains a section on derived sets and ordinal numbers, a short note on Jordan measure, and a description of the process of constructing the Borel sets. The second discusses Borel measure in n-dimensional Euclidean space R_n. The third discusses various approaches to the definition of the family of Lebesgue measurable sets and outlines the approaches of Lebesgue and Carathéodory to measure and measurability.

The first section of the introduction is mainly devoted to an illustration of the use of ordinal numbers and a statement of the principle of transfinite induction. The Borel classes B_α are defined in the usual way by transfinite induction.

The second booklet gives the process for defining a completely additive nonnegative measure on the family B of Borel sets in R_n by transfinite induction starting from the usual definition of volume for n-dimensional cubes. The validity of the process is proved first for bounded sets and extended to all of B in the natural way.

The third booklet begins with a proof that every Jordan-measurable set is a Borel set minus some subset of a Borel set of measure zero. This leads to the definition of the family L of Lebesgue measurable sets. The σ-field N of subsets of Borel sets of measure zero is defined first; L is the set of all sums $X \cup Y$ with X in B and Y in N. It is proved that Borel measure can be extended from B to L by setting $m(Z) = m(X)$ if $Z = X \cup Y$ with X in B and Y in N. The properties of Lebesgue measure are easily proved by this technique from the corresponding theorems on Borel measure. This definition of measure and measurability is then connected with the methods of Lebesgue and Carathéodory which use inner and outer measures.

The format of the booklets suggests that they are intended for use as the introduction of a formal course in measure theory. The selection and arrangement of material are somewhat different from those used in the introductions to measure theory in general use. The exposition is clear and coherent. *M. M. Day* (Urbana, Ill.).

Jessen, Børge. Abstract theory of measure and integration. V. Mat. Tidsskr. B. **1942**, 43–53 (1942). (Danish) [MF 13589]

Jessen, Børge. Abstract theory of measure and integration. VI. Mat. Tidsskr. B. **1944**, 28–34 (1944). (Danish) [MF 13597]

Jessen, Børge. Abstract theory of measure and integration. VII. Mat. Tidsskr. B. **1944**, 35–36 (1944). (Danish) [MF 13598]

[Part IV appeared in Mat. Tidsskr. B. **1939**, 7–21 (1939); these Rev. **1**, 110.] This continues a series of expository articles. Parts V and VI cover, respectively, Lebesgue theory in Euclidean spaces and in spaces of infinitely many dimensions. Part VII treats complex-valued functions. *W. Feller* (Ithaca, N. Y.).

Sparre Andersen, Erik. Content and measure in product sets. Mat. Tidsskr. B. **1944**, 19–23 (1944). (Danish) [MF 13595]

The author considers product spaces of arbitrary ordered collections of spaces; the collections may have any power. It is shown that a Peano content or a measure in the product space can be defined if a content or measure is given for each component space. *W. Feller* (Ithaca, N. Y.).

Buch, Kai Rander. A minimum problem in abstract space. Mat. Tidsskr. B. **1945**, 30–34 (1945). (Danish) [MF 14259]

Let $\varphi(A)$ and $\psi(A)$ be two measure functions defined on the same completely additive family of sets in a space E. In the case where E is a Euclidean space the Neyman-Pearson method of "testing the statistical hypothesis that a certain random variable has the distribution function $\varphi(A)$ against the simple alternative that the distribution is $\psi(A)$" leads to the problem of determining a set for which $\varphi(A)$ equals a given constant k and $\psi(A)$ is as large as possible. The author proves the existence of a solution to the problem. If $\xi(A)=\varphi(A)-c\psi(A)$, it is shown that there exists at least one set A' on which $\xi(A)$ attains its minimum. This set A' is a solution of the problem for the particular value $k=\varphi(A')$. The set A' is completely characterized by the property that for all subsets A of A' the inequality $\varphi(A)\geqq c\psi(A)$ holds, while the inequality is reversed for all subsets of $E-A'$. *W. Feller* (Ithaca, N. Y.).

Erdös, Paul. Some remarks on the measurability of certain sets. Bull. Amer. Math. Soc. **51**, 728–731 (1945). [MF 13610]

The author treats sets defined by simple geometric properties. The principal tool is the Lebesgue density theorem. The following results are established. (1) Let E be any closed set in n-dimensional Euclidean space and let E_r be the set of points whose distance from E is r ($r>0$). Then E_r has measure 0. (2) Let A be any set of measure 0 on the positive real axis, and let E_A be the set of points whose distance from E is in A. Then E_A has measure 0. More generally, if S is an arbitrary measurable set, E_S is measurable. (3) Let M be the set of points for each of which there is more than one closest point in E. Then M has measure 0. (4) Let x be any point in the complement of E, and let $\varphi(x)$ be the set of points in E closest to x. Then $\sum\varphi(x)$, the summation being taken over all x not in E, has measure 0. (5) Let M_k be the set of points for which $\varphi(x)$ contains k points not all in a $(k-2)$-dimensional Euclidean subspace. Then in 2-dimensional Euclidean space M_3 has Hausdorff dimension 0 and is in fact denumerable. It is stated that M_3 has Hausdorff dimension 0 and is denumerable in n-dimensional Euclidean space and it is conjectured that M_k has Hausdorff dimension $n+1-k$. (6) Let E be bounded. Then E_r has finite $(n-1)$-dimensional measure. *G. B. Price* (Lawrence, Kan.).

Tolstoff, G. Sur les points de densité des ensembles linéaires mesurables. Rec. Math. [Mat. Sbornik] N.S. **10(52)**, 249–264 (1942). (Russian. French summary) [MF 12834]

Let E be any measurable set on the x-axis, CE its complement, and let $E(a, b)$ denote the intersection of E with the interval (a, b). Lebesgue's theorem on the points of density asserts that

$$(*) \qquad \lim_{h\to 0} |h|^{-1}\,\text{meas}\,CE(x, x+h)=0$$

for almost every point $x\in E$. The author investigates the problem of the rapidity with which (*) takes place. His first result is as follows. (i) For every function $\varphi(h)$ defined for $0<h\leqq h_0$, positive and increasing there and tending to 0 with h, there is a perfect nondense set P of positive measure such that

$$|h|^{-1}\,\text{meas}\,CP(x, x+h)<\varphi(|h|)$$

for almost every $x\in P$ and for all h satisfying $|h|\leqq\delta=\delta(x)$. (ii) Similarly, for every function $\varphi(h)$ with properties just described there is a perfect nondense set P of positive measure such that

$$|h|^{-1}\,\text{meas}\,CP(x, x+h)>\varphi(|h|)$$

for all x and for all $|h|$ sufficiently small (uniformly in x). A connection of the function φ in (i) with the lengths of the complementary intervals of P is seen in the following result. (iii) If $u_1, u_2, \cdots$ are the lengths of the intervals contiguous to a perfect set P, and if $u_n\leqq q^n$ $(0<q<1)$, then for every positive number α we have

$$|h|^{-1}\,\text{meas}\,CP(x, x+h)<|h|^{\alpha}$$

for almost every $x\in P$ and for $|h|\leqq\delta(x)$. *A. Zygmund.*

Young, L. C. A lemma in the theory of surfaces. J. London Math. Soc. **19**, 209–212 (1944). [MF 14569]

The fact that there is no upper bound for the diameter of surfaces whose boundaries and areas are given causes the lack of compactness which hampers the calculus of variations in the study of surfaces. The author proves, however, that such an upper bound (depending on η) does exist if the surfaces concerned are modified by the following two operations: (i) addition of a closed surface S' of area not exceeding η; (ii) removal of a closed surface S'' of area not less than that of S'. The proof, given only for the case of polyhedra, is contained in the following lemma. Let S be a polyhedron, of area N, represented on the unit square, and let the boundary of S lie within a sphere of radius a. Then, given $\epsilon>0$, there exists a closed polyhedron S', of area not exceeding ϵN, such that $S+S'=S_0+S''$, where S'' is closed and of area at least that of S' and S_0 and S' lie within a concentric sphere of radius $a\exp(1/\epsilon)$. In addition, several variants of this lemma are given. *G. B. Price.*

Gomes, A. Pereira. Semicontinuous functions and the Darboux property. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. 15, 18 pp. (1945) = Anais Fac. Ci. Pôrto 29, no. 4. (Portuguese. French summary) [MF 13298]

A space (V) is one in which a closure operation, denoted by a bar, has been defined with the properties (1) $\bar{O}=O$, (2) $X \subset Y$ implies $\bar{X} \subset \bar{Y}$, (3) $X \subset \bar{X}$. Here O is the null set. If we add (4) $\bar{\bar{X}}=\bar{X}$, we obtain a space (F). These axioms are weaker than the ones due to Kuratowski. A real-valued (including $\pm\infty$) function $f(x)$ is upper semi-continuous if for all sets X, $f(\bar{X}) \subset [\alpha, \infty]$, where $\alpha = \inf f(X)$. Also $f(x)$ is called semi-continuous if it is both upper and lower semi-continuous. In a space (V) continuity implies semi-continuity but not conversely. In a Kuratowski space the concepts are identical. The author proves that the space (V) is connected if and only if it possesses the generalized Darboux property. The latter is defined as follows: every semi-continuous function on (V) assumes all values between any two. The result constitutes an extension of a theorem of Sierpiński. *E. R. Lorch* (New York, N. Y.).

Marcouchewitch, A. Sur le prolongement par continuité. Rec. Math. [Mat. Sbornik] N.S. 16(58), 43–58 (1945). (Russian. French summary) [MF 13012]

The author defines an element of a continuous function to be a sphere in Euclidean n-space R^n plus a function $P=f(p)$ which is single-valued and continuous in this sphere (P is also assumed to be a point of R^n). The set of elements is subjected to a uniqueness condition which is equivalent to the statement that, if the spheres of two elements have a common point, then the values of these elements either coincide at all points of the common part or coincide over no domain belonging to the common part. By continuation throughout a domain G a function is defined, and the author gives conditions in order that the function be single-valued in G (monodromy theorem).

Let p_0 be a point in a neighborhood of which $f(p)$ is single-valued and continuous (except at p_0). If the set of limit values of $f(p)$ at p_0 is not a single point of R^n (closed by addition of the point at infinity), then p_0 is called an essential singular point. Conditions are given in order that a given set of points constitute a set of exceptional values of a function $f(p)$ in the neighborhood of an essential singularity.

Finally, the author generalizes a theorem of Hadamard on the inversion of transformations which are locally homeomorphic. *D. C. Spencer* (Stanford University, Calif.).

***Nef, Walter. Ein Satz über die Differenzierbarkeit der Funktionen von beschränkter Schwankung in topologischen Räumen.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 201–208, Füssli, Zürich, 1945.

The author, extending the theorem that a function of bounded variation is differentiable almost everywhere, proves in slightly greater generality a theorem which reduces in a compact, separable, regular space X to the following. For each P in X let a P-sequence be a sequence of open sets $\{\mu_i\}$ such that $\bar{\mu}_{i+1} \subset \mu_i$ and $\prod_{i<\omega}\mu_i = P$. Let m be a measure function defined on open sets in X such that (1) $m(\mu)$ is positive if μ is nonempty; (2) if $\mu=\sum_i \mu_i$, then $m(\mu) \leqq \sum_{i<\omega} m(\mu_i)$; (3) for each P-sequence $\{\mu_i\}$, $\lim_{i\to\omega} m(\mu_i)=0$; (4) for each $\epsilon>0$ and each μ, there is a μ' for which $m(\mu')<\epsilon$ and $\mu' \supset (\bar{\mu}-\mu)$. Let α be a completely additive function defined on open sets and of bounded variation, that is, such that there is a K with $\sum_i |\alpha(E_i)| < K$ if $\{E_i\}$ is a finite sequence of open sets or differences of open sets. Then for almost every P in X the derivative $D\alpha/Dm = \lim_{i\to\omega} \alpha(\mu_i)/m(\mu_i)$ exists and has the same value for every P-sequence. *M. M. Day* (Urbana, Ill.).

Poor, Vincent C. On the Hamilton differential. Bull. Amer. Math. Soc. 51, 945–948 (1945). [MF 14466]

The paper contains two theorems about various definitions of the differential of a point function $\phi(P)$ in a vector space of finite dimensionality. If $\phi(P+\lambda dP)$ admits a derivative with respect to λ at $\lambda=0$ for a fixed P and every dP, then ϕ is said to have the Hamilton differential $\phi'(P, dP)$ at P (also called the Gâteaux differential). Let P_λ, Q_λ, R_λ denote variable points which approach P as $\lambda \to 0$. The first theorem states that, if the Hamilton differential exists throughout a neighborhood of P and is such that (1) $\phi'(R_\lambda, \lambda^{-1}(Q_\lambda - P_\lambda)) \to \phi'(P, dP)$ whenever $\lambda^{-1}(Q_\lambda - P_\lambda) \to dP$, and (2) ϕ' is a distributive function of dP, then also $\lambda^{-1}\{\phi(Q_\lambda) - \phi(P_\lambda)\} \to \phi'(P, dP)$. The conclusion is equivalent to the statement that $\phi'(P, dP)$ is a differential in the sense of Rainich [Amer. J. Math. 46, 71–94 (1924), in particular, p. 78]. The proof does not require hypothesis (2), but merely that ϕ' is homogeneous of degree one in dP. [The author is apparently unaware that the Hamilton differential has the latter property automatically. Since, as the author states, the Rainich differential has the property (2), the distributivity of ϕ' can be made a part of the conclusion rather than of the hypothesis. The proof given is valid only when ϕ is a real scalar function, but by applying the argument to the components separately the proof can be extended to functions with values in a finite dimensional real vector space. Infinite dimensional spaces are not considered.] *A. E. Taylor* (Los Angeles, Calif.).

Linés Escardó, E. The method of the arbitrary function in calculus of probability. Revista Mat. Hisp.-Amer. (4) 5, 3–52, 63–72 (1945). (Spanish) [MF 12992]

The author considers the following problem, attributed to Poincaré [Calcul des Probabilités, 2d ed., Gauthier-Villars, Paris, 1912]: do there exist subsets C of the unit interval for which $I = \int_C f(x)dx / \int_0^1 f(x)dx = m(C)$ independently of $f(x)$? [The sets for which this is true are obviously the sets of measure zero or one.] It is shown that, for fixed $f(x)$, I is approximately $m(C)$ for all sets C which are sufficiently "close" to being "uniformly dense." By extending Lebesgue measure, the author constructs a set C for which I is $\frac{1}{2}$ for every summable function $f(x)$. *D. Blackwell.*

Lévy, Paul. Propriétés intrinsèques des fonctions, et intégrales de Stieltjes. Bull. Soc. Math. France 69, Communications et Conférences 5–9 (1941). [MF 13246]

The function $\Phi(h)$ being continuous and monotonic for $h \geqq 0$, with $\Phi(0)=0$ and $\Phi(\infty)=\infty$, and having the continuous inverse $\varphi(h)$, a function $f(x)$ on $a \leqq x \leqq b$ has bounded Φ-variation [see L. C. Young, Math. Ann. 115, 581–612 (1938)] if the expression $\sum_{k=1}^{n} \Phi|f(x_k) - f(x_{k-1})|$ is bounded for all subdivisions $a \leqq x_0 < x_1 < \cdots < x_n = b$, it being assumed that $x-0$ and $x+0$ are distinct with $x-0 < x+0$. The principal theorem is that the smallest class of functions invariant under continuous one-to-one transformation of the variable for which the Lipschitz condition $\Phi|f(x+\Delta x) - f(x)| \leqq \Delta x$ is satisfied is the class of continuous functions of bounded Φ-variation. The paper speculates on possible necessary conditions for the existence of the Riemann-Stieltjes integral $\int u dv$ based on the existence theorem of L. C. Young

[loc. cit., p. 598] involving Φ-variations of u and v, but neglects to notice that the limit involved in Young's definition is that of successive subdivision, and not the "norm" limit used by Riemann. *T. H. Hildebrandt.*

Ayer, Miriam C. On convergence in length. Proc. Nat. Acad. Sci. U. S. A. **31**, 261–266 (1945). [MF 13294]

Let $\tau(u)$ be the vector $(x(u), y(u), z(u))$, $u\varepsilon I=(\alpha\leqq u\leqq\beta)$. Each function x, y, z is continuous and the length $L(I,\tau)=L(\tau)$ is defined as l.u.b. $\sum\{\tau(u'')-\tau(u')\}$. If x_n, y_n, z_n converge uniformly, τ_n is said to converge uniformly, $\tau_n\to\tau_0\{\mathfrak{U}\}$; τ_n converges strongly in length, $\tau_n\to\tau_0\{\mathfrak{SL}\}$, if and only if $\tau_n\to\tau_0\{\mathfrak{U}\}$ and $L(\tau_n-\tau_0)\to 0$. If Δ is a subinterval of I the interval function $L(\Delta,\tau)$ is nonnegative, completely additive and continuous, and can be extended to a completely additive function $L(E,\tau)$ of Borel sets E. The Lebesgue decomposition then gives $L(E,\tau)=L_a(E,\tau)+L_s(E,\tau)$, where $L_a(E,\tau)\geqq\int_E|\tau'|\geqq 0$ is AC, and $L_s(E,\tau)\geqq 0$ is singular. If $f(u)$ is BV on I then $f=f_a+f_s$, where f_a is AC and f_s is singular. This is the normal Lebesgue decomposition of f, and is uniquely determined if it is agreed that the singular part vanishes at the left end of I. This decomposition of x, y and z gives $\tau=\tau_a+\tau_s$; τ_a and τ_s give rise to functions of Borel sets, $L(E,\tau_a)$, $L(E,\tau_s)$.

If $\tau(u)$ is BV and continuous on I then $L_a(E,\tau)=L(E,\tau_a)$ and $L_s(E,\tau)=L(E,\tau_s)$ for every Borel set $E\subset I$. This leads to new results and makes it possible to simplify known results. If τ_n is continuous and BV on I then $\tau_n\to\tau_0\{\mathfrak{SL}\}$ if and only if $\tau_{na}\to\tau_{0a}\{\mathfrak{SL}\}$, $\tau_{ns}\to\tau_{0s}\{\mathfrak{SL}\}$. If $\tau_1(u)$ and $\tau_2(u)$ are continuous and BV on I then $L(\tau_1+\tau_2)=L(\tau_1)+L(\tau_2)$ if and only if $L(\tau_{1a}+\tau_{2a})=L(\tau_{1a})+L(\tau_{2a})$ and $L(\tau_{1s}+\tau_{2s})=L(\tau_{1s})+L(\tau_{2s})$. If $\tau=(u, f(u), 0)$, $f(u)$ continuous and BV, then $L(\tau)\leqq(\beta-\alpha)+V(f)$, equality holding if and only if f is singular. This implies that if $f(u)$ is monotone and singular $L(\tau)=(\beta-\alpha)+f(\beta)-f(\alpha)$, which is a generalization of the fact that the length of the Cantor function on $0\leqq x\leqq 1$ is 2. If $\tau_1(u)$ and $\tau_2(u)$ are BV and continuous on I then $L\{(\tau_1+\tau_2)/2\}\leqq\{L(\tau_1)+L(\tau_2)\}/2$, the Steiner inequality. If τ_1 and τ_2 are AC on I then the equality holds if and only if $\tau_1'/|\tau_1'|=\tau_2'/|\tau_2'|$ almost everywhere on E, the set for which $u\varepsilon I$, $|\tau_1'|>0$, $|\tau_2'|>0$.

Several inequalities are given for the case in which $\tau_1(u)$, $\tau_2(u)$ are BV and continuous. There are results concerning convergence such as: if $\tau_n\to\tau_0\{\mathfrak{U}\}$, then $\int_E|t_n-t_0|\to 0$, where $E=E[u\varepsilon I, |\tau_0'|>0, \liminf|\tau_n'|=0]$ and t_n is the unit tangent vector $\tau_n'/|\tau_n'|$.

If the functions x_n, y_n, z_n converge in variation or in measure τ_n is said to converge in variation or in measure: $\tau_n\to\tau_0\{\mathfrak{B}\}$, $\tau_n\to\tau_0\{\mathfrak{M}\}$. If $\tau_n=(u, y_n(u), z_n(u))$ and τ_0 is AC then $\tau_n\to\tau_0\{\mathfrak{L}\}$ if and only if $\tau_n\to\tau_0\{\mathfrak{B}\}$ and $\int_I|t_n-t_0|\to 0$. If τ_n is BV and continuous and τ_0 is AC, then $\tau_n\to\tau_0\{\mathfrak{SL}\}$ if and only if $\tau_n\to\tau_0\{\mathfrak{B}\}$ and $|\tau_n'-\tau_0'|\to 0\{\mathfrak{M}\}$ on I. If τ_n represents polygons inscribed in the curve represented by $\tau_0(u)$, τ_0 BV and continuous, then $\tau_n\to\tau_0\{\mathfrak{SL}\}$ if and only if $\tau_n\to\tau_0\{\mathfrak{U}\}$ and τ_0 is AC. *R. L. Jeffery* (Kingston, Ont.).

Federer, Herbert. The Gauss-Green theorem. Trans. Amer. Math. Soc. **58**, 44–76 (1945). [MF 12772]

The validity of the Gauss-Green formula connecting the integral of the partial derivative of a function over a region and the integral of the function over the boundary of the region is extended, with no specialization of the function, in such a way that the region can be an open n-dimensional set whose boundary B has $\Phi(B)$ finite (where Φ is an $(n-1)$-dimensional measure function which is the Carathéodory linear measure function if $n=2$) and satisfies one additional condition on B which is shown to be satisfied if $n=2$. Whether this extra condition on B is automatically satisfied if $\Phi(B)<\infty$ and $n>2$ is an open question. The starting point is the definition of the external normal of a subset A of n-space at a point x. Before reading this paper one acquainted with Carathéodory's approach to measure theory needs only to have read some of the definitions used in two of the author's papers [same Trans. **55**, 420–437, 438–456 (1944); these Rev. **6**, 44, 45] and to refer to a paper by A. P. Morse and the reviewer [same Trans. **55**, 236–305 (1944); these Rev. **5**, 232]. The point θ of E_n is defined by $\theta=(0, \cdots, 0)$. When $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ the notation $x\cdot y=\sum_j x_j y_j$ is used. Also K_x^r denotes the set of $z\varepsilon E_n$ such that $|z-x|<r$ for $x\varepsilon E_n$, $r>0$. It is said that u points into S at x if and only if $S\subset E_n$, $x\varepsilon E_n$, $u\varepsilon E_n$, $|u|=1$ and

$$\lim_{r\to 0+}|H_r-S|/|H_r|=0,$$

where H_r is the hemisphere of K_x^r for which $u>0$; u is called an external normal to S at x if and only if u points into E_n-S at x and $-u$ points into S at x. It is shown that, if u and v are external normals of S at x, then $u=v$. In case S has an external normal at x, this normal is designated by $\nu(S,x)$; otherwise $\nu(S,x)=\theta$. Hence with $\nu_j(S,x)$ the jth coordinate of $\nu(S,x)$ it is seen that $\nu_j(S,x)=0$ whenever S has no external normal at x. Thus

$$-\infty<\int f(x)\nu_j(S,x)d\Phi x<\infty$$

(integration being over the whole space) does not imply that the external normal of S exists anywhere. One of the theorems is that if $A\subset E_n$ and f is the function such that $f(x)=\nu(A,x)$ for $x\varepsilon E_n$, then f is a Borel measurable function (that is, the counter image of every closed set is a Borel set).

The main purpose is the investigation of the Gauss-Green formula

$$\int_A D_j f(x)dx=\int f(x)\nu_j(A,x)d\Phi x. \tag{1}$$

Here A is an open set of E_n and $D_j f$ is the partial derivative of f in the direction of the jth unit vector. The assumption is always made that both integrals are finite, the boundary of A has finite Φ measure, and f is absolutely continuous within the closure of A along almost all lines in the direction of the jth unit vector. Specifically, for $j=1, \cdots, n$ and $S\subset E_n$ the class $\Omega_j(S)$ is defined as follows: $f\varepsilon\Omega_j(S)$ if and only if f is a numerically valued function, S is a subset of the domain of f,

$$-\infty<\int f(x)\nu_j(S,x)d\Phi x<\infty,\quad -\infty<\int_S D_j f(x)dx<\infty,$$

and there is a set $V\subset E_{n-1}$ such that $|V|=0$ and

$$f(y_1, \cdots, y_{j-1}, b, y_j, \cdots, y_{n-1})-f(y_1, \cdots, y_{j-1}, a, y_j, \cdots, y_{n-1})=\int_a^b D_j f(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1})dt$$

whenever $y\varepsilon E_{n-1}-V$, $a<b$ and $(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1})\varepsilon S$ for $a\leqq t\leqq b$. For $n=2$ the statement of the theorem is very simple. If A is a bounded open subset of E_2, B is the boundary of A, $\Phi(B)<\infty$, j is either 1 or 2 and $f\varepsilon\Omega_j(A+B)$, then the equation (1) holds. If $n>2$ there is an extra condition

on the boundary which requires more notation than seems advisable to put into a review. For $n=2$, a consequence is the following form of Cauchy's theorem. If A is a bounded open set of complex numbers, B is the boundary of A, $\Phi(B)<\infty$, f is conformal at each point of A (that is, $f'(x)\varepsilon E_2$ for $x\varepsilon A$) and f is continuous at each point of B with respect to $A+B$, then $\int f(x)\nu(A,x)d\Phi x=0$.

One not interested in the specific problem at hand may still want to read the proof of the first theorem of the paper, which is about transformation of integrals and requires no knowledge of the special notations and techniques of the rest of the paper. If φ is such a measure over A that A is expressible as a countable sum of φ measurable sets of finite measure, if ψ is such a measure over the metric space B that closed subsets of B are ψ measurable and every ψ measurable set is contained in a Borel set of equal ψ measure, if g is such a function on A to B that the set on which $g(x)\varepsilon Y$ is φ measurable for every closed set $Y\subset B$, if u is such a φ measurable function that $0\leqq u(x)<\infty$ for φ almost all x in A and

$$\int N(g,X,y)d\psi y=\int_X u(x)d\varphi x$$

for every φ measurable set $X\subset A$ (where $N(g,X,y)$ is the number of times in X that g takes on the value y), then

$$\int f(y)N(g,X,y)d\psi y=\int_X f\{g(x)\}u(x)d\varphi x$$

whenever X is a φ measurable subset of A and f is such a ψ measurable function that $-\infty\leqq f(y)\leqq\infty$ for ψ almost all y in B. *J. F. Randolph* (Oberlin, Ohio).

Bouligand, Georges. Dependance entre deux fonctions réelles d'un point du plan et application à une methode d'Ossian Bonnet. Rev. Sci. (Rev. Rose Illus.) **79**, 302–306 (1941). [MF 13322]

The function $Y(x,y)$ is called a function of $X(x,y)$ if $Y(x,y)$ is constant on each locus $X(x,y)=$constant. This definition is applied to functions of class (H): $f(x,y)$ is of class (H) if there is a function $g(x,y)$ such that $X=f(x,y)$, $Y=g(x,y)$ determines a homeomorphism between a region of the (x,y) plane and one of the (X,Y) plane. The main result is an analytic condition that $Y(x,y)$ is a function of $X(x,y)$ when both are given merely of class (H) in a region R. To each point $m_0:(x_0,y_0)$ of R are attached certain sequences (unspecified) from which exclusively the values of $x-x_0$ and $y-y_0$ are taken. On the continuum of R containing m_0, a sequence of points is taken converging to m_0 and tangent to a half-line m_0t_0. [Reviewer's remark: because of the exclusive values above, not all directions may be possible.] A further restriction on the sequence is that it shall give limiting values, denoted by X_x and X_y, to the ratios $(X-X_0)/(x-x_0)$ and $(X-X_0)/(y-y_0)$. [The reviewer is uncertain as to the arguments of X here, since if $X(x,y)$ is meant, then equation $aX_x+bX_y=0$ below would imply that $X_x=X_y=0$.] Letting a,b be the components of a vector along the half-line, then $m_0;a,b$; X_x and X_y are called a concomitant system [cf. same Rev. **79**, 246–249 (1941); these Rev. **7**, 74]. The author states, either without proof or with a very short proof not clear to the reviewer, that a necessary and sufficient condition for the dependence of $Y(x,y)$ on $X(x,y)$ is that the set of all relations of the form $aX_x+bX_y=0$ implies the set of all relations $aY_x+bY_y=0$. [Later he uses the same a,b in both equations.] As an application, a geometric result of Ossian Bonnet [reference to G. Julia, Cours de Géométrie, Gauthier-Villars, Paris, 1941, pp. 470–472] is obtained and extended somewhat, with lessened conditions of differentiability of the functions involved. Another application is to give an analogue of the Jacobian for the case of a function given merely as of class (H). *A. B. Brown* (Flushing, N. Y.).

Theory of Functions of Complex Variables

***Ríos, Sixto. Conferencias sobre la Representación Analítica de Funciones. [Lectures on the Analytic Representation of Functions.]** C. Bermejo, Madrid, 1945. 137 pp. (Spanish)

In these lectures (written in collaboration with L. Vigil) the author covers a wide variety of topics in the representation of analytic functions of a complex variable: Runge's theorems; analytic continuation by overconvergence and by rearrangement (he constructs, among other examples, a "universal" series of polynomials which can be rearranged to converge uniformly to any prescribed analytic function in any desired region); Mittag-Leffler, Borel and Painlevé expansions; analytic continuation by summation of series; representation of functions by Laplace integrals and by Dirichlet, factorial, interpolation and Lambert series.

R. P. Boas, Jr. (Providence, R. I.).

Uspensky, J. V. On the arithmetico-geometric means of Gauss. II. Math. Notae **5**, 57–88 (1945). (Spanish) [MF 14100]

Uspensky, J. V. On the arithmetico-geometric means of Gauss. III. Math. Notae **5**, 129–161 (1945). (Spanish) [MF 14578]

Continuation of the exposition begun in Math. Notae **5**, 1–28 (1945); these Rev. **7**, 7. These parts consider the arithmetico-geometric mean of two complex numbers and its connections with modular functions. *R. P. Boas, Jr.*

Teissier du Cros, F. Sur la convergence d'une série entière dont le terme général à sa partie réelle bornée en deux points de la circonférence-unité. C. R. Acad. Sci. Paris **219**, 44–45 (1944). [MF 14512]

Valiron, Georges. Sur l'approximation des nombres réels et un théorème de M. Teissier du Cros. C. R. Acad. Sci. Paris **219**, 45–47 (1944). [MF 14513]

Let $S=\sum\gamma_n z^n$. If the general term of the real part of S is bounded for $z=e^{i\theta_1}$ and $z=e^{i\theta_2}$ and if $(\theta_2-\theta_1)/\pi=\omega$ is algebraic and irrational, then the series S converges for $|z|<1$. This result of the first paper is extended to a larger class of numbers ω in the second paper, which also contains a related result on the zeros of certain entire functions.

R. Salem (Cambridge, Mass.).

Delange, Hubert. Sur les suites de fractions rationnelles à zéros et pôles réels. C. R. Acad. Sci. Paris **221**, 371–373 (1945). [MF 14677]

The following is typical of several results stated by the author without proof. Let $R_n(z)$ denote a sequence of rational fractions all of whose zeros and poles are real, and let $\phi(n)$ be a positive function of the integer n. The zeros and poles of $R_n(z)$ lie between fixed values a and b. Denote by $A_n z^{p_n}$, $B_n z^{q_n}$ the leading terms of the numerator and denominator, and by $\nu_n(x)$ the number of zeros of $R_n(z)$ which are at most equal to x minus the number of poles not greater

than x. In the first place, whenever $\nu_n(x)/\phi(n)$ remains bounded, independently of x and n, in order that the sequence $\{\phi(n)\log|R_n(z)|\}^{-1}$ is convergent outside $[a, b]$ it is necessary and sufficient that (1) the sequence $M_n=\{\int_a^t \nu_n(u)du\}/\phi(n)$ converges on $[a, b]$ to a limit function M, (2) $(p_n-q_n)/\phi(n)$ has a finite limit h, (3) $\{\phi(n)\log|A_n/B_n|\}^{-1}$ has a finite limit λ. In the second place, whenever $\nu_n(x)\geqq 0$ for all n and x the conditions are the same, except that in (1) M_n converges to M for all values of t for which M is continuous and at the ends of the finite interval. *M. S. Robertson* (New Brunswick, N. J.).

Sagatelan, W. W. Sur l'intégrale du type de Cauchy-Stieltjes à limites infinies. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 23–32 (1944). (Russian. French summary) [MF 15189]

The integral in question is of the form

$$\Phi(z)=(2\pi i)^{-1}\int_{-\infty}^{\infty}(x-z)^{-1}dF(x),$$

where $F(x)$ is a complex valued function of bounded variation in every finite interval and the integral converges absolutely. It is called a Cauchy-Stieltjes integral if its limit as z approaches the real axis from above is almost everywhere $F'(x)$. If $F(x)$ is absolutely continuous, we have an integral of Cauchy-Lebesgue type or a Cauchy-Lebesgue integral, respectively. The author studies the boundary values of these integrals and the classes of functions which they represent; this continues work of Privaloff on integrals along other paths [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 195–197 (1940); see these Rev. **2**, 181, where further references are given]. The following theorems are typical. For almost all real x_0,

$$\Phi(z)\to(2\pi i)^{-1}\int_{-\infty}^{\infty}(x-x_0)^{-1}dF(x)\pm\tfrac{1}{2}F'(x_0)$$

as $z\to x_0$ along any nontangential path [the author's statement inadvertently omits the "non-"], where the upper and lower signs correspond to the upper and lower half planes. It follows that $\Phi(z)\equiv 0$ for $\Im(z)<0$ is necessary and sufficient for $\Phi(z)$ to be a Cauchy-Stieltjes integral; in that case $F(x)$ must be absolutely continuous. A function $f(z)$ analytic in the upper half-plane is represented by the Cauchy-Lebesgue integral of its boundary function if and only if it belongs to H_1 [that is, $\int_{-\infty}^{\infty}|f(x+iy)|dx$ has an upper bound independent of y] and its boundary function $f(x)$ is such that $f(x)/(x+i)\in L(-\infty, \infty)$ [new proof of a theorem of Kryloff, Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 95–138 (1939); these Rev. **1**, 308]. If $f(x)/(x+i)\in L$ and $f(x)$ satisfies a Lipschitz condition of order less than 1, then the limit function of the Cauchy-Lebesgue integral formed with $f(x)$ satisfies a Lipschitz condition of the same order.

R. P. Boas, Jr. (Providence, R. I.).

Gattegno, Caleb. Nouvelle démonstration d'un théorème de M. Ostrowski sur la représentation conforme à la frontière. Proc. Math. Phys. Soc. Egypt **2**, no. 1, 1–4 (1941). [MF 14109]

The author gives a new demonstration of a theorem of Ostrowski concerning the boundary behavior of a conformal mapping [Prace Mat.-Fiz. **44**, 371–471 (1936), in particular, p. 441]. *M. H. Heins* (Cambridge, Mass.).

Dufresnoy, Jacques. Sur un théorème d'Ahlfors et son application à l'étude de la représentation conforme. C. R. Acad. Sci. Paris **220**, 424–427 (1945). [MF 13955]

In conformal mapping the existence of an angular derivative can be studied by a method due to the reviewer [Acta Soc. Sci. Fenn. Nova Ser. A. **1**, no. 9 (1930)]. The reviewer's two inequalities are sharpened and proved under more general conditions; in particular, the second inequality is proved for unsymmetric regions. Using these results and a theorem of Ostrowski [Prace Mat.-Fiz. **44**, 371–471 (1936)] the author arrives at a new sufficient condition.

L. Ahlfors (Zurich).

Ferrand, Jacqueline. Extension d'une inégalité de M. Ahlfors. C. R. Acad. Sci. Paris **220**, 873–874 (1945). [MF 14073]

Continuing the paper reviewed above, the author gives simplified proofs and further generalizations of the second inequality. With their help she obtains a remarkably sharp sufficient condition for the existence of the angular derivative. *L. Ahlfors* (Zurich).

Ferrand, Jacqueline. Sur la déformation analytique d'un domaine. C. R. Acad. Sci. Paris **221**, 132–134 (1945). [MF 14244]

The author formulates an analogue of a theorem of Marčenko [C. R. (Doklady) Acad. Sci. URSS (N.S.) **6**, 289–290 (1935)] concerning the conformal mapping of "nearly circular" regions. She considers the conformal mapping of a plane simply-connected region Δ which possesses the following property: there exist a point z_0 and polygons Δ_0 and Δ_1 such that (i) $z_0\in\Delta_0\subset\Delta\subset\Delta_1$, and (ii) the polygon Δ_0 is carried into the polygon Δ_1 by a transformation of the form $z|z_0+(1+\epsilon)(z-z_0)$, where ϵ is real and positive.

M. H. Heins (Cambridge, Mass.).

Bieberbach, Ludwig. Über Schlitzabbildungen durch rationale Funktionen. Deutsche Math. **5**, 272–273 (1940). [MF 14332]

Remarks of a polemical character. *M. H. Heins.*

Kufareff, P. P. On one-parameter families of analytic functions. Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 87–118 (1943). (Russian. English summary) [MF 11648]

To each t of the segment $a\leqq t\leqq b$ let there correspond a domain $G(t)$ in the plane of the complex variable z and a function $F(z, t)$ which is regular in $G(t)$, and let $G^*(t_0)$ denote the kernel (in the sense of Carathéodory) of the sequence of domains $G(t)$ as t approaches t_0. The author investigates the existence of the partial derivatives $\partial F(z, t)/\partial t$ for $z\in G^*(t_0)$. In the first part of the paper cases are discussed in which $\partial F(z, t)/\partial t$ exists for almost all t, $a\leqq t\leqq b$; in the second part, cases in which the derivative exists at a point. Moreover, formulas for the partial derivative are given which are analogous to the well-known equation of Löwner for functions regular and schlicht in the interior of the unit circle. In particular, the author obtains a generalization of Löwner's equation which he proves is valid for almost all t in $a\leqq t\leqq b$. *D. C. Spencer.*

Robertson, M. S. Univalent power series with multiply monotonic sequences of coefficients. Ann. of Math. (2) **46**, 533–555 (1945). [MF 14117]

Let $s_n(z)=1+z+z^2+\cdots+z^n$, $S_n(z)=s_0(z)+\cdots+s_n(z)$. Sufficient conditions are given for the univalency in $|z|\leqq 1$

of a linear combination $c_0S_0(z)+\cdots+c_nS_n(z)$, where the c_k are nonnegative and do not all vanish. For instance, $c_nS_n(z)+\cdots+c_mS_m(z)$ will be univalent whenever the sequence $(c_{n+2}, c_{n+3}, \cdots, c_m, 0, 0, \cdots)$ is monotonic of order 2. Thus $c_nS_n(z)+c_{n+1}S_{n+1}(z)+c_{n+2}S_{n+2}(z)$ is always univalent. [Compare Egerváry, Math. Z. **42**, 221–230 (1937).] The paper also contains a new proof and an extension of Fejér's theorem [Acta Litt. Sci. Szeged **8**, 89–115 (1936)] that a power series $\sum_1^\infty a_nz^n$ with real coefficients is univalent in $|z|<1$ if the sequence (a_n) is monotonic of order 4 [monotonic of order 3 is sufficient; see G. Szegö, Duke Math. J. **8**, 559–564 (1941); these Rev. **3**, 76]. *W. W. Rogosinski.*

Golusin, G. M. Some estimations of derivatives of bounded functions. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 295–306 (1945). (Russian. English summary) [MF 14585]

The author considers only functions $f(z)$ which are regular and satisfy $|f(z)|\leqq 1$ in $|z|<1$. In the first part of the paper he obtains precise bounds for $|f'(z)|$ under various additional restrictions on f. For example, if $f(z)=c_0+c_nz^n+c_{n+1}z^{n+1}+\cdots$, then

$$|f'(z)|\leqq (nr^{n-1}/(1-r^{2n}))(1-|f(z)|^2), \qquad |z|=r,$$

and equality occurs only for the functions

$$e^{i\theta}(z^n+\alpha)/(1+\bar{\alpha}z), \qquad |\alpha|\leqq 1.$$

Next, the radius of starlikeness is given when f is of the form $c_mz^m+c_{m+n}z^{m+n}+c_{m+n+1}z^{m+n+1}+\cdots$, $m\geqq 1$, $n\geqq 1$, and also when f is subjected to the further restriction that $f(z)\neq 0$ in $0<|z|<1$. The author obtains a bound for $|f'(0)|$ when f takes the same value at p given points of $|z|<1$, and applies this result to obtain the radius of p-valence of f in terms of the quantities $|f(0)|$ and $|f'(0)|$ [theorem of Dieudonné, Ann. Sci. École Norm. Sup. (3) **48**, 247–358 (1931)]. Finally, given a sequence of real numbers $\lambda_m, \lambda_{m+1}, \cdots, \lambda_m>0, \lambda_k\geqq 0$ for $k\geqq m+1$, it is shown that, if $f(z)=c_mz^m+c_{m+1}z^{m+1}+\cdots$, then $\sum_{k=m}^\infty \lambda_k|c_k|r^k\leqq \lambda_mr^m$ for $r<\rho$, where ρ is a root of the equation $2\sum_{k=m+1}^\infty \lambda_kr^k=\lambda_mr^m$. [In the case $m=0$, $\lambda_k=1$ $(k=0, 1, \cdots)$ we have $\rho=\frac{1}{3}$; this is a theorem of Bohr.] *D. C. Spencer.*

Ríos, Sixto. The theorems of Picard. Revista Mat. Hisp.-Amer. (4) **5**, 152–163 (1945). (Spanish) [MF 15205]

An expository lecture.

Besicovitch, A. S. A note on integral functions. J. London Math. Soc. **19**, 207 (1944). [MF 14566]

The author gives a new and simple proof for the well-known theorem that an integral function $f(z)$ of order less than one cannot be bounded in a half-plane without being bounded in the whole plane. *M. S. Robertson.*

Mursi, M. An identity in integral functions. Proc. Math. Phys. Soc. Egypt **1**, no. 4, 14–16 (1940). [MF 14107]

Let $f(z)$ and $g(z)$ be entire functions with zeros a_n, b_n and orders ρ, σ, respectively. If k, exceeding the exponents of convergence of $\{a_n\}$ and $\{b_n\}$, can be chosen so that the circles of radii $|a_n|^{-k}$ about a_n and of radii $|b_m|^{-k}$ about b_m never overlap, then there exist entire functions $F(z)$, $G(z)$, of orders at most $\max(\rho, \sigma)$, such that $f(z)F(z)+g(z)G(z)\equiv 1$. The interest of the result lies in the possibility of restricting the orders of $F(z)$ and $G(z)$ under a suitable restriction on the zeros of $f(z)$ and $g(z)$. *R. P. Boas, Jr.*

Carleman, T. Sur la détermination d'un fonction analytique par certaines valeurs moyennes de ses dérivées. Ark. Mat. Astr. Fys. **32B**, no. 4, 7 pp. (1945). [MF 15096]

Let $M(f(z))=\int_\Omega f(z)d\omega_s/\int_\Omega d\omega_s$, where $f(z)$ is analytic in the region Ω and ω is an additive nonnegative set function of bounded variation. The author's problem is to determine conditions under which an entire function $f(z)$, at most of order 1 and type 0, is determined by the relations $M(f^{(\nu)}(z))=a_\nu$, $\nu=0, 1, 2, \cdots$. His result is that, if $|a_\nu|^{1/\nu}\to 0$ $(\nu\to\infty)$, there is a unique $f(z)$, given by

$$2\pi i f(z)=\int_C \frac{g(w)e^{z/w}}{wH(w)}dw,$$

where $g(w)=\sum_0^\infty a_\nu w^\nu$, $H(w)=\sum_0^\infty M(z^\nu)w^{-\nu}/\nu!$, and C is a circle containing the origin and all the zeros of $H(w)$.

R. P. Boas, Jr. (Providence, R. I.).

Okada, Yoshitomo. On the representations of functions in the theory of interpolation. Tôhoku Math. J. **49**, 119–132 (1942). [MF 14703]

The author generalizes the Newton and Gauss interpolation series by replacing the arithmetic progression of real interpolation points by a set of several arithmetic progressions lying on several lines in the complex plane. He expresses the remainder of the series as a contour integral and, by making suitable estimates, obtains restrictions on the growth of an entire function $f(z)$ which will ensure that the corresponding interpolation series will converge to $f(z)$.

R. P. Boas, Jr. (Providence, R. I.).

Gontcharoff, W., et Gontcharoff, M. Sur une série d'interpolation procedant suivant certaines fonctions rationnelles. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 3–22 (1944). (Russian. French summary) [MF 15188]

The series is

$$(1) \qquad c_0+\sum_{n=1}^\infty c_n\frac{z-a_1}{1-a_1z}\cdots\frac{z-a_n}{1-a_nz},$$

where $\{a_n\}$ is a real increasing sequence with limit 1 and $\sum(1-a_n)$ diverges. The series (1) converges in a region $D(u)$: $\Re\{(z+1)/(z-1)\}<u$. Necessary and sufficient conditions are given for the expansion when $\lim_{n\to\infty}(1-a_n)n^\sigma=\omega$, $0<\sigma\leqq 1$, $0<\omega<\infty$. For example, if $\sigma<1$ and $u>0$, the class of functions having the expansion (1) is the class of all functions regular and bounded in $D(u)$; if $u<0$, the class consists of all functions regular in $D(u)$ and satisfying a growth condition near $|z|=1$ (of the order of $|1-z|^{\frac{1}{2}(\sigma-1)/\sigma}$ if $\sigma<1$). The series was studied under more restrictive conditions by Lammel [Časopis Pěst. Mat. Fys. **66**, 57–61 (1936); **67**, 103–108 (1938); Monatsh. Math. Phys. **46**, 321–323 (1938)]. *R. P. Boas, Jr.* (Providence, R. I.).

Brunk, H. D. Dirichlet series meromorphic in a half-plane. Amer. J. Math. **67**, 486–504 (1945). [MF 13937]

The author continues his researches arising out of Mandelbrojt's extension to Dirichlet series of Hadamard's multiplication theorem [H. D. Brunk, Duke Math. J. **12**, 1–21 (1945); these Rev. **6**, 268]. He considers functions $f(s)=f(\sigma+i\tau)$ which are "M" in a half-plane $\sigma>\sigma_1$. A function is M in $\sigma>\sigma_1$ in particular if it is "bounded except for singularities" in $\sigma>\sigma_1$, but the class of M functions is more general than this. The author applies his results to Dirichlet series which possess a finite abscissa A of absolute convergence. Let H be the abscissa of holomorphism and H^*

the ultimate abscissa of holomorphism, that is, the greatest lower bound of numbers ξ such that $f(s)$ is holomorphic for $\sigma>\xi$ and $|\tau|>\tau_0(\xi)$. He proves (1) if $f(s)=\sum a_n e^{-\lambda_n s}$ is M in $\sigma>\sigma_0$ for some $\sigma_0<H-D$, where $D=\limsup n^{-1}\log n$, then $H^*\geqq H-D$. (2) If $f(s)$ is M in $\sigma>\sigma_1$ for some $\sigma_1<A-D$, then $H\geqq A-D$. In particular, (3) if $f(s)$ is M in $\sigma>\sigma_0$ for some $\sigma_0<A$, and if $D=0$, then $H^*=H=A$. The author applies his results to obtain information about the singularities of functions defined by Dirichlet series for which $D=0$. *L. S. Bosanquet* (London).

Ahlfors, Lars V. Die Begründung des Dirichletschen Prinzips. Soc. Sci. Fenn. Comment. Phys.-Math. **11**, no. 15, 15 pp. (1943). [MF 14633]

The author gives an elegant development of Dirichlet's principle, using ideas related to those of Nikodym [J. Math. Pures Appl. (9) **12**, 95–108 (1933)] and Weyl [Duke Math. J. **7**, 411–444 (1940); these Rev. **2**, 202]. He applies his method to the construction of Abelian differentials on Riemann surfaces with prescribed singularities and periods. Let F be a closed Riemann surface. If ω is a differential on F and in a local coordinate system $\omega=pdx+qdy$, we set $\omega^*=-qdx+pdy$, $d\omega=(q_x-p_y)dxdy$, $\|\omega\|^2=\iint_F(p^2+q^2)dxdy$. Let α be a differential on F with the prescribed singularities and periods and such that $d\alpha=0$. We assume that the singularities are isolated point singularities and that there is a differential β such that $d\beta=0$, $\|\alpha+\beta^*\|<\infty$, and such that the coefficients of $\alpha+\beta^*$ are of class C^1 everywhere on F. The problem is to find a function u of class C^2 on F such that $d\omega=d\omega^*=0$, where $\omega=\alpha+du$.

Following Dirichlet's principle, we look for a differential of the form $\omega=\alpha+du$ which minimizes $\|\omega+\beta^*\|$. Let $\{\omega_n\}$ be a minimizing sequence. A classical argument shows that $\{\omega_n\}$ is convergent in the above norm. In any parameter circle $|z|\leqq 1$ containing no singularities let $\omega_n=p_n dr/r+q_n d\varphi$, $z=re^{i\varphi}$. The Fourier coefficients of q_n on any circle $|z|=r$ converge. A variational argument shows that the limiting values of these coefficients have the form $a_k r^k$. This enables the author to prove that q_n converges in the mean to a harmonic function. From here it is easy to show that $\{\omega_n\}$ converges to a differential having the required properties. *P. C. Rosenbloom* (Providence, R. I.).

Garnier, René. Sur le problème de Riemann-Hilbert. C. R. Acad. Sci. Paris **221**, 276–278 (1945). [MF 14497]

Let C be a simple closed curve in the complex z-plane and let $A(z)$ be a square matrix defined on C. The Riemann-Hilbert problem for C and $A(z)$ is the problem of finding matrices $\Phi(z)$ and $\Psi(z)$, of the same order as $A(z)$, analytic in the interior and exterior, respectively, of C, and determining boundary values satisfying $\Phi=A\Psi$ on C. The problem was solved by G. D. Birkhoff [Proc. Amer. Acad. Arts Sci. **49**, 521–568 (1913)] under certain analyticity conditions concerning C and $A(z)$. In the present paper the method of Birkhoff is extended to obtain the result for a broad class of continuous matrices $A(z)$. *E. F. Beckenbach* (Los Angeles, Calif.).

Vekua, N. The problem of Riemann with discontinuous coefficients for several unknown functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 1–10 (1944). (Georgian. Russian summary) [MF 14600]

Cf. Muschelišvili and Vekua, Trav. Inst. Mat. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 1–46 (1943); these Rev. **6**, 272; and the following review. The author states that a Russian translation will appear elsewhere.

Vekua, N. On a linear boundary value problem of Riemann with discontinuous coefficients for a system of analytic functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 473–482 (1944). (Georgian. Russian summary) [MF 14606]

According to the summary, the author uses his results on singular integral equations [same Bull. **5**, 125–134 (1944); these Rev. **7**, 207] to generalize results of I. N. Vekua [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 109–139 (1942); these Rev. **6**, 123] and B. V. Chvedelidze [same Bull. **4**, 289–296 (1943); these Rev. **6**, 149].

Differential Equations

Chazy, Jean. Sur la théorie des centres. C. R. Acad. Sci. Paris **221**, 7–10 (1945). [MF 14233]

Consider a differential equation of the form $dx/X=dy/Y$, where $X=y+X_2$, $Y=-x+Y_2$, X_2 and Y_2 being power series in x and y which contain no constant or linear terms. Poincaré [J. Math. Pures Appl. (4) **1**, 167–244 (1885)] sought to construct an integral of the form $F=x^2+y^2+F_3+F_4+\cdots$, where F_n is a homogeneous polynomial of degree n. In the determination of the typical function F_n there arises a constant C_n. Poincaré showed that, if all these C's are zero, the singular point (0, 0) is a center and otherwise it is a focus. The author determines the nature of the focus more precisely. Specifically, if $C_k=0$ for $k<n$, and $C_n\neq 0$, ρ tends to zero as $\theta^{-1/(n-2)}$ as θ increases. (Here ρ and θ are polar coordinates in the (x, y)-plane.) The note also contains a simplification of the proof of the convergence of the series $F=x^2+y^2+F_3+F_4+\cdots$. *L. A. MacColl.*

Andronow, A., et Bautin, N. Stabilisation de route d'un avion neutre par autopilote ayant un servomoteur à vitesse constante et à zone d'insensibilité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 143–146 (1945). [MF 12702]

The methods of a previous paper [same C. R. (N.S.) **43**, 189–193 (1944); these Rev. **6**, 194] are extended to the case when there is a symmetric "dead zone" in which the stabilizing servo motor does not respond, and similar criteria for stability are established. *W. Kaplan.*

Hulthén, Lamek. Some integral theorems for eigenfunctions of a discrete spectrum. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **15**, no. 22, 215–223 (1945). [MF 14282]

By use of Green's theorem the author shows that, if $\psi(0)=\psi(\infty)=0$, $\psi''+(E+v(x))\psi=0$ and $\psi_0''+E\psi_0=0$, then

$$\psi'(0)=\int_0^\infty \exp\{-x\sqrt{(-E)}\}v(x)\psi(x)dx$$

and

$$\lim_{x\to\infty}\exp\{-x\sqrt{(-E)}\}\{\psi\cdot\sqrt{(-E)}-\psi'\}=2\int_0^\infty \sinh(x\sqrt{(-E)})v(x)\psi(x)dx.$$

Use is made of the fact that $\psi_0=\exp\{\pm x\sqrt{(-E)}\}$. These results may be used to test the accuracy of approximate

wave functions since they relate the behavior of ψ at $x=0$ and $x=\infty$. Generalizations are indicated and a numerical example is given. *H. Feshbach* (Cambridge, Mass.).

Patry, Jean. Le théorème de Fuchs et les équations linéaires à coefficients périodiques. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 118–122 (1942). [MF 14203]

This note is concerned with differential equations of the form

$$(*) \qquad \sum_{m=0}^{n}(e_m+f_m e^{-ix}+g_m e^{ix})d^m u/dx^m=0.$$

If $f_n\neq 0$, or if $f_k=0$ and $e_n\neq 0$, the transformation $z=e^{ix}$ reduces the equation to the form

$$z^n u^{(n)}+z^{n-1}P_{n-1}(z)u^{(n-1)}+\cdots+P_0(z)u=0,$$

where the P's are functions which are holomorphic in the neighborhood of $z=0$. The classical Fuchs theory is then applied to obtain n solutions of the form $u=\sum_{k=0}^{\infty}a_k z^{\rho+k}=\sum_{k=0}^{\infty}a_k e^{i(\rho+k)x}$. If $g_n\neq 0$, or if $g_k=0$ and $e_n\neq 0$, the transformation $z=e^{-ix}$ can be used in a similar manner to obtain n solutions of the form $u=\sum_{k=0}^{\infty}a_k e^{i(\rho-k)x}$. It is shown that, in order that this procedure shall give series which converge for real values of x, it is necessary that the absolute values of the roots of the equation $g_n z^2+e_n z+f_n=0$ are both less than unity, or both greater than unity. *L. A. MacColl.*

Patry, Jean. Une méthode numérique pour résoudre les équations linéaires à coefficients périodiques. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 122–126 (1942). [MF 14204]

Continuing his studies of the differential equation (*) of the note reviewed above, the author here employs a method of solution which is an adaptation of a method often used for the solution of Mathieu's equation. Writing $u=\sum_{k=-\infty}^{\infty}D_k e^{i(\rho+k)x}$, we obtain a system of recurrence relations for the D's, of the form

$$A(\mu+k)D_{k-1}+B(\mu+k)D_k+C(\mu+k)D_{k+1}=0.$$

The ratios D_k/D_0 are then expressed in terms of continued fractions. Substitution of the values of D_1/D_0 and D_{-1}/D_0 in the equation $A(\mu)D_{-1}+B(\mu)D_0+C(\mu)D_1=0$ then gives an equation which determines μ (or determines a parameter in the differential equation, if the value of μ is determined by other considerations). Examining the convergence of the process, the author arrives at the result that the method gives a solution which is valid for real values of x, provided the roots of the equation $f_n+e_n z+g_n z^2=0$ satisfy the conditions $|z_1|<1$, $|z_2|>1$. *L. A. MacColl.*

Patry, Jean. Sur la résolution des équations différentielles linéaires à coefficients périodiques. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 225–229 (1942). [MF 14209]

A few additional remarks on the subjects of the two notes reviewed above. *L. A. MacColl* (New York, N. Y.).

Mendes, M. Sur une équation aux dérivées partielles du second ordre. Ann. Fac. Sci. Univ. Toulouse (4) **6**, 83–105 (1943). [MF 15173]

An abstract of this paper was given by the author in C. R. Acad. Sci. Paris **212**, 112–114 (1941); these Rev. **2**, 365. *F. G. Dressel* (Durham, N. C.).

Soudan, Robert. Expression nouvelle de la densité de balayage d'un corps homogène. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 26–27 (1945). [MF 14227]

Let there be given a body of uniform density δ bounded by a surface S. It is tacitly assumed that S is sufficiently smooth to permit the use of the formulas of classical potential theory. Let P be an arbitrarily chosen fixed point of space and let R be a variable point of S. Let X be the solution of the interior Dirichlet problem for S and the boundary values $4\pi\delta\overline{RP}^2/3$. The author shows that if the sweeping out process is applied to the mass within S the resulting distribution on S has at R the density

$$\omega(R)=\tfrac{1}{3}\delta\frac{d}{dn}\overline{RP}^2-\frac{1}{4\pi}\frac{d}{dn}X(R),$$

where d/dn indicates the interior normal derivative of S at R. *F. W. Perkins* (Hanover, N. H.).

Brelot, Marcel. Sur le problème de Dirichlet. C. R. Acad. Sci. Paris **221**, 654–656 (1945). [MF 15152]

Using conformal mapping, Evans considered the Dirichlet problem for plane regions with multiple boundary points [The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems, Amer. Math. Soc. Colloquium Publ., vol. 6, New York, 1927; see chapter V]. The problem in space has been studied by Perkins [Trans. Amer. Math. Soc. **38**, 106–144 (1935)], by de la Vallée Poussin [Les nouvelles méthodes de la théorie du potentiel, Actual. Sci. Indus. no. 578, Hermann, Paris, 1937; see pp. 38–46] and by Kappos [Ann. Scuola Norm. Super. Pisa (2) **11**, 43–63 (1942)].

In this paper the author outlines further work on this topic. He considers the extended Dirichlet problem for a domain Ω of the compact space $\bar{R}_\tau$ of $\tau\geqq 2$ dimensions [see the following review]. He is particularly interested in the extension of some of the ideas relative to discontinuous boundary values developed for the simple problem in an earlier paper [Acta Litt. Sci. Szeged **9**, 133–153 (1939); these Rev. **1**, 121]. *F. W. Perkins* (Hanover, N. H.).

Brelot, M. Sur le rôle du point à l'infini dans la théorie des fonctions harmoniques. Ann. Sci. École Norm. Sup. **61**, 301–332 (1944). [MF 13773]

In dealing with problems concerning harmonic and subharmonic functions in a Euclidean space R_τ of $\tau\geqq 2$ dimensions the domain of definition is often restricted to be bounded. The author proposes to study some of these problems in the space $\bar{R}_\tau$ obtained by adjoining to R_τ the point at infinity $\mathfrak{R}_\tau$. Considerable use is made of material in two earlier papers by the author [Acta Litt. Sci. Szeged **9**, 133–153 (1939); Bull. Sci. Math. (2) **65**, 72–98 (1941); these Rev. **1**, 121; **7**, 15].

The author discusses first the behavior at infinity of a function u subharmonic in an unbounded domain ω of R_τ. Among the results obtained are a condition for the existence of a harmonic majorant for u in the neighborhood of the point at infinity and a condition that u be a potential due to negative masses. Next he considers the case in which the domain ω of $\bar{R}_\tau$ contains $\mathfrak{R}_\tau$ and defines as subharmonic in ω any function u which meets the usual requirements in $\omega-\mathfrak{R}_\tau$, is not positively infinite at $\mathfrak{R}_\tau$, is upper semi-continuous and such that $u(\mathfrak{R}_\tau)$ is not greater than the integral mean of u on the surface of a sphere with center chosen arbitrarily and radius then chosen sufficiently large. A similar extension of the definition of harmonic functions and a generalized form of Poisson's integral are also given.

The author applies his definitions in a discussion of several topics connected with subharmonic functions, including polar sets and thin sets [cf. Brelot, J. Math. Pures Appl. (9) **19**, 319–337 (1940); Bull. Sci. Math. (2) **68**, 12–36 (1944); these Rev. **3**, 47; **7**, 15]. He also considers H. Cartan's results on families of potentials [C. R. Acad. Sci. Paris **214**, 994–997 (1942); these Rev. **5**, 146]. Especial attention is given to the Dirichlet problem for a domain in $\bar{R}_\tau$ and to the representation of subharmonic functions as potentials in $\bar{R}_\tau$. *F. W. Perkins* (Hanover, N. H.).

Brelot, M. Sur l'approximation et la convergence dans la théorie des fonctions harmoniques ou holomorphes. Bull. Soc. Math. France **73**, 55–70 (1945). [MF 14175]

Let $\bar{R}_\tau$ be the space obtained by adjoining the point at infinity to the Euclidean space R_τ of $\tau \geqq 2$ dimensions [cf. the preceding review]. Let E be a closed set in $\bar{R}_\tau (E \neq \bar{R}_\tau)$, CE the complement of E and E^* the frontier of E. Let u be any function finite and continuous on E^*. A necessary and sufficient condition that it is possible to approximate to u on E^* within ϵ by a function harmonic in an open domain containing E is that all frontier points of E are stable (that is, that CE is not thin at any point of E^*). This theorem is related to a result given by Walsh [J. Reine Angew. Math. **159**, 197–209 (1928)]. Another theorem is obtained by letting α be a closed set contained in CE, Δ the sum of those domains which form a part of the open set CE and which contain at least one point of α and u a function continuous on E^*. A necessary and sufficient condition that it is possible to approximate to u (within ϵ on E^*) by a function harmonic in $C\alpha$ is that E^* is the frontier of $C\Delta$ and that every point of E^* is stable for $C\Delta$. [See the paper by Deny reviewed below.]

The following results concerning convergence make use of ideas previously developed by the author [see, in particular, Acta Litt. Sci. Szeged **9**, 133–153 (1939); these Rev. **1**, 121]. The abbreviation p.p.-μ (presque partout-μ) is used to indicate that a possible exceptional set is of harmonic measure zero. Let Ω be an open set in $\bar{R}_\tau$ such that $C\Omega$ is nonpolar, and let f be a function defined on the frontier Ω^*. If f is resolvent (that is, if $\underline{H}_f = \bar{H}_f = H_f$) and if $H_f = 0$ then $f = 0$, p.p.-μ. If $|f_n| \leqq g$ (where each f_n and g are resolvent) and if $H_{f_n} \to 0$, then on the set on which $\lim f_n$ exists, $\lim f_n = 0$, p.p.-μ. Let u_n, v_n be harmonic and have common bounds in Ω; let u_n and v_n approach boundary values φ_n and ψ_n, respectively, p.p.-μ on $\alpha \subset \Omega^*$; as $n \to \infty$ let $\varphi_n \to \varphi$ and $\psi_n \to \psi$, p.p.-μ on α. (a) If $\alpha = \Omega^*$ and if $\varphi = \psi$, p.p.-μ, then u_n and v_n converge as $n \to \infty$ to a common harmonic function U, and their pseudo-limits at any irregular boundary point converge to that of U. (b) If u_n and v_n have a common limit U, then $\varphi = \psi$, p.p.-μ on α. Let E be a compact set in $\bar{R}_\tau (E \neq \bar{R}_\tau)$ and let the functions u_n be harmonic in open sets containing E and have common bounds. If the sequence of u_n's converges at the stable frontier points of E, then it converges everywhere on E^*, and if two such sequences have common limits on the stable points, they have common limits throughout E.

The final section of the paper deals with similar convergence properties of holomorphic functions. The author cites related work by Montel, Khintchine, Lavrentieff [M. Lavrentieff, Sur les fonctions d'une variable complexe représentables par des séries de polynomes, Actual. Sci. Indus. no. 441, Hermann, Paris, 1936, in particular, chap. III, 8, 9]. *F. W. Perkins* (Hanover, N. H.).

Deny, Jacques. Sur l'approximation des fonctions harmoniques. Bull. Soc. Math. France **73**, 71–73 (1945). [MF 14176]

This paper is devoted to material closely related to the first part of the paper reviewed above. The theory is formulated for a Euclidean space R_τ of $\tau \geqq 2$ dimensions, but both the author and Brelot comment on the possibility of extending the discussion so as to replace R_τ by the space $\bar{R}_\tau$ considered by Brelot.

The author defines as elementary harmonic functions relative to the point A expressions of the form

$$\Phi_n^A(M) = H_n(M)/r^{2n+\tau-2},$$

$n = 0, 1, 2, \cdots$, where $r = \overline{AM}$ and H_n is a harmonic polynomial. In the special case in which $\tau = 2$, $\Phi_0^A = \log(1/r)$. In this case we may choose Φ_n^A as $r^{-n}\cos n\theta$ or $r^{-n}\sin n\theta$. [These last formulas are those clearly intended by the author; they are given inaccurately in the paper.]

Let E be a compact set in R_τ. Its complement CE is the sum of a finite or denumerably infinite set of domains $D_0 + D_1 + D_2 + \cdots$; let A_p be a point of D_p, $p = 0, 1, 2, \cdots$. The principal result of the paper may now be stated as follows. In order that every function F continuous on E and harmonic in the interior (if any) of E may be uniformly approached by linear combinations of the functions $\Phi_n^{A_p}$ it is necessary and sufficient that E have no unstable frontier points. *F. W. Perkins* (Hanover, N. H.).

Ettorre, Maria Laura. Calcolo della frequenza più bassa di una membrana vibrante con contorno fisso avente la forma di ellisse. Rend. Accad. Sci. Fis. Mat. Napoli (4) **13**(82), 12 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. **140**. [MF 11509]

An approximation to the lowest eigenvalue of a vibrating elliptic membrane is obtained by introducing polar coordinates into the equation $\Delta u + \lambda u = 0$ and by putting

$$u(\rho, \theta) = \tfrac{1}{2}u_0(\rho) + \sum_{k=1}^{\infty} \{u_k(\rho)\cos k\theta + v_k(\rho)\sin k\theta\}.$$

A. Weinstein (Toronto, Ont.).

Biçadze, A. Boundary value problems for a system of linear differential equations of elliptic type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 761–770 (1944). (Georgian. Russian summary) [MF 14611]

The system has the form

$$\Delta u_i + \sum_{k=1}^{n} \{A_{ik}(x, y)\frac{\partial u_k}{\partial x} + B_{ik}(x, y)\frac{\partial u_k}{\partial y} + C_{ik}(x, y)u_k\} = 0,$$

where Δ is the Laplacian. The author states in his summary that he applies the method of I. N. Vekua [same Bull. **1**, 29–34, 181–186, 497–500 (1940); **4**, 385–392 (1943); these Rev. **2**, 201; **3**, 128; **6**, 154] to reduce his problems to systems of singular integral equations of the type discussed by N. I. Mushelišvili and N. P. Vekua [same Bull. **3**, 987–994; **4**, 207–214; Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 1–46 (1943); these Rev. **5**, 269; **6**, 272].

Laichtman, D. L. Radiative cooling of the air-layer near the earth. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 89–93 (1945). (Russian. English summary) [MF 14576]

A study is made of the variation in temperature with the altitude in the air layer near the earth during a clear night.

It is assumed that the coefficient of heat conduction is a power function of the altitude, and that the earth is cooled by radiation. The problem is reduced to the solution of a differential equation with given boundary conditions. The theoretical results are compared with observational data.

H. P. Thielman (Ames, Iowa).

Rott, N. Die Aufwärmung unbegrenzter Körper durch eine stetig wirkende Wärmequelle. Schweiz. Arch. Angew. Wiss. Tech. **11**, 164–174, 212–216 (1945). [MF 13766]

The problems treated are idealizations of problems in the temperatures generated by the welding of thin plates and rods. Let $T(x, y, t)$ denote the temperature of a thin plate of infinite extent at a point (x, y) at time t. A formula is derived for the temperature $T(x, y, t)$ induced by a constant point-source of heat which moves at a uniform rate along the positive half of the x-axis. Linear transfer of heat is assumed at the faces of the plate. As a special case the faces may be insulated. The author notes simple rules for modifying the temperature formula in the case of a fixed source so as to obtain the formula in the case of the moving source. Problems of variable and steady temperatures in an infinitely long wire, with a continuous heat source at one point or over a segment, are treated first. Corresponding problems for the plate are then solved, with some observations about problems in three or more dimensions. The formulas are reduced to forms that are practical for computations and the general nature of the temperature variations is noted. *R. V. Churchill* (Ann Arbor, Mich.).

Géhéniau, J. Sur les formules d'Hadamard relatives aux équations aux dérivées partielles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 591–607 (1943). [MF 13856].

The paper contains a résumé of the following two sets of formulas: (1) Hadamard's formulas for the solution of Cauchy's problem in the case of the general second order linear partial differential equation of normal hyperbolic type; (2) the formulas for the differentiation with respect to a parameter u of integrals (of various multiplicities) whose domains of integration are subject to given infinitesimal transformations of the form $dx^i = X^i(x_1, \cdots, x_n, u)du$. The latter set of formulas is well known in the theory of integral invariants, and they are introduced here because indicated differentiations of just this sort occur in the Hadamard formulas. In other words, any practical computation based on formulas (1) could in general only be carried out with the help of formulas (2). The paper appears to contain no essentially new results. *D. C. Lewis.*

De Donder, Th. Simplification de la méthode d'intégration d'Hadamard. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30**, 763–765 (1943). [MF 13866]

It is pointed out that the formulas of Hadamard for the solution of Cauchy's problem for the general second order linear partial differential equation of normal hyperbolic type can be somewhat simplified by means of a simple device which replaces the given Cauchy problem by a modified problem in which the Cauchy data vanish identically. A similar, if not identical, device has been previously used for simplifying the formulas of Riemann in the two dimensional case [cf. R. Courant and D. Hilbert, Methoden der Mathematischen Physik, vol. 2, Springer, Berlin, 1937, p. 318, footnote]. *D. C. Lewis* (Durham, N. H.).

Martin, Monroe H. The rectilinear motion of a gas. II. Amer. J. Math. **67**, 407–410 (1945). [MF 12920]

Continuing previous work [same J. **65**, 391–407 (1943); these Rev. **4**, 278], the author considers the rectilinear motion of a gas under the assumption that no heat energy is transferred by radiation or conduction (so that each element of gas is deformed adiabatically). The problem of determining velocity and pressure as functions of position and time, when their initial values are given, is reduced to a Cauchy problem for a second order partial differential equation, to which the method of separation of variables is applicable. *W. Kaplan* (Ann Arbor, Mich.).

Integral Equations

***Janet, Maurice. Équations intégrales et applications à certains problèmes de la physique mathématique.** Mémor. Sci. Math. no. 101 and 102. Gauthier-Villars, Paris, 1941. 151 pp.

A presentation of the fundamental results of the theory of linear integral equations with applications to ordinary differential equations of the second order, based to a large extent on the Hilbert-Courant theory. Table of contents: Quadratic forms; linear oscillations; vibrating strings; Green's function; systems of linear equations; integral equations with degenerate kernel; orthogonal functions; infinite sets of functions, equal continuity; Fredholm formulas; remarks on Fredholm formulas; integral equation with symmetric kernel; existence of characteristic value; direct definition of nth characteristic value; Hilbert's theorem; complete character of a system of characteristic functions; new extremal properties of characteristic numbers; conditions so that characteristic values are all positive; correlation of the two extremal problems; generalizations, singular integral equations; extension to n variables; exercises. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Bateman, H. Two integral equations. Proc. Nat. Acad. Sci. U. S. A. **31**, 196–200 (1945). [MF 12798]

The author discusses solutions of the integral equations

$$f(s) = \int_0^\infty k(s, t)F(t)dt,$$

$$f(s) = F(s) - x\int_0^\infty k(s, t)F(t)dt,$$

where $k(s, t) = g(s, t) - cg(s, a-t)$; $g(s, t)$ is a real continuous kernel. *O. Szász* (Cincinnati, Ohio).

Parodi, Maurice. Équations intégrales de Fredholm et calcul symbolique. C. R. Acad. Sci. Paris **220**, 870–872 (1945). [MF 14072]

It is shown that the Laplace transform can be used to solve the integral equation $F(y) = \int_0^\infty K(y, z)f(z)dz$, provided that the kernel satisfies an integral equation of the form

$$g(x)^{\nu+1} \exp\{-zg^\mu(x)\} = \int_0^\infty e^{-xy}K(y, z)dy, \quad \nu, \mu > 0.$$

H. Pollard (New Haven, Conn.).

Parodi, Maurice. Application du calcul symbolique à la résolution d'une équation de Volterra dont le noyau n'appartient pas au groupe du cycle fermé. C. R. Acad. Sci. Paris **221**, 18–19 (1945). [MF 14237]

By using a modified form of the unilateral Laplace trans-

form, the author solves the integral equation

$$\int_0^t J_0((t^2-x^2)^{\frac{1}{2}})f(x)dx=g(t)$$

for the function $f(x)$, where $g(t)$ is assumed known and J_0 is the Bessel function of the zeroth order of the first kind. [For the modified form of the transform see van der Pol, Physica 1, 521–530 (1934) and J. P. Schouten, Physica 2, 75–80 (1935).] *A. E. Heins* (Lafayette, Ind.).

Täcklind, Sven. Elementare Behandlung vom Erneuerungsproblem für den stationären Fall. Skand. Aktuarietidskr. 27, 1–15 (1944). [MF 14144]

Täcklind, Sven. Fourieranalytische Behandlung vom Erneuerungsproblem. Skand. Aktuarietidskr. 28, 68–105 (1945). [MF 14155]

These papers treat the integral equation of renewal theory,

$$\varphi(t)=p(t)+\int_0^t \varphi(t-x)p(x)dx,$$

where $p(t)$ is a frequency function (derivative of a distribution function). It is well known that under very general assumptions $\varphi(t)$ converges in the mean to $1/m$, where m is the first moment of $p(t)$. This result is usually deduced by Fourier analysis. In the first paper the same conclusion is reached from simple elementary considerations. The more refined questions of proper convergence and of asymptotic developments [Feller, Ann. Math. Statistics 12, 243–267 (1941); these Rev. 3, 151] cannot be treated with such simple methods. They are investigated in the second paper using the theory of Laplace transforms. Instead of the Tauberian theorem of Haar used by Feller, the author uses a theorem of Titchmarsh [Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937, theorem 74] which turns out to be better suited for the purpose. Feller's asymptotic developments are thus slightly improved. In particular, convergence of $\varphi(t)$ as $t\to\infty$ is proved replacing the condition that $p(t)$ is of bounded variation by the more convenient condition that $p(t)\in L_2$. *W. Feller.*

Reissner, Eric. Solution of a class of singular integral equations. Bull. Amer. Math. Soc. 51, 920–922 (1945). [MF 14461]

The equation

$$g(x)=\pi^{-1}\int_{-1}^{1} f(\xi)\left\{(\xi-x)^{-1}+\sum_{n=0}^{N} c_n(\xi-x)^{2n+1}\right\}d\xi,$$

where the coefficients c_n are given constants, is a generalization of a well-known equation in the theory of airfoils. By the familiar method of orthogonal functions its solution is reduced to that of a system of $2N+2$ linear algebraic equations. *M. Golomb* (Philadelphia, Pa.).

Vekua, N. On the theory of systems of singular integral equations with discontinuous coefficients. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 125–134 (1944). (Georgian. Russian summary) [MF 14601]

We consider a system of the form

$$(1)\quad \sum_{\beta=1}^{n} A_{\alpha\beta}(t_0)\rho_\beta(t_0)+\sum_{\beta=1}^{n}\frac{B_{\alpha\beta}(t_0)}{\pi i}\int_L \frac{\rho_\beta(t)dt}{t-t_0}+\sum_{\beta=1}^{n}\frac{1}{\pi i}\int_L K_{\alpha\beta}(t_0,t)\rho_\beta(t)dt=f_\alpha(t_0),$$

$\alpha=1, 2, \cdots, n$, where L is a sum of simple closed smooth nonintersecting plane contours bounding certain bounded regions; $A_{\alpha\beta}(t_0)$, $B_{\alpha\beta}(t_0)$, $K_{\alpha\beta}(t_0)$ and $f_\alpha(t_0)$ are functions given on L, having, in general, discontinuities of the first kind at certain points $a_1, \cdots, a_\nu$ of L, while $A_{\alpha\beta}(t_0)$, $B_{\alpha\beta}(t_0)$ satisfy Hölder conditions in the interior of each arc $a_\alpha a_{\alpha+1}$ ($\alpha=1, 2, \cdots, \nu$; $a_{\nu+1}=a_1$), and Lipschitz conditions at the ends of these arcs; $K_{\alpha\beta}(t_0, t)$, $f_\alpha(t_0)$ satisfy Hölder conditions on each arc $a_\alpha a_{\alpha+1}$; the unknown functions $\rho_\alpha(t_0)$ satisfy Hölder conditions everywhere on L except possibly at $a_1, \cdots, a_\nu$, where they may have poles of order not higher than the first. In the first sum on the left of (1) the integrals are Cauchy principal values.

From the author's summary.

Bose, S. N. On an integral equation associated with the equation for hydrogen atom. Bull. Calcutta Math. Soc. 37, 51–61 (1945). [MF 13782]

Studying the Schrödinger equation of the hydrogen atom in momentum space, the author seems to be unaware of previous work on that problem [V. Fock, Z. Physik 98, 145–154 (1935); E. T. Copson, Proc. Roy. Soc. Edinburgh. Sect. A. 61, 260–272 (1943); these Rev. 5, 8]. He deduces an integral equation for the wave function in momentum space. This integral equation was stated without proof by Fock; the proof given by the author is essentially identical with that given by Copson and attributed to E. T. Whittaker. The method adopted in this paper for solving the integral equation is different from that used by Fock, but does not appear to offer any additional advantages. The results are identical. *A. Erdélyi* (Edinburgh).

Vekua, I. N. On the Prandtl integro-differential equation. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 143–150 (1945). (Russian. English summary) [MF 14040]

A solution is given of the integrodifferential equation of Prandtl in the theory of a wing of finite span:

$$(8\pi/mb(x))\Gamma(x)-\int_{-a}^{a}(\Gamma'(t)/(t-x))dt=4\pi V\alpha(x),$$

where $b(x)$ is a chord of the profile, x is the distance of the profile from the axis of symmetry of the span, $m=5.5$, α is a certain angle, a is the half span and V is the velocity of flow at infinity. This equation is singular, since it involves integration in the sense of Cauchy principal values. The problem is reduced to a regular Fredholm integral equation. Profiles of the form

$$b(x)=b_0(1-x^2/a^2)\frac{1+\nu_1x^2a^{-2}+\cdots+\nu_nx^{2n}a^{-2n}}{1+\mu_1x^2a^{-2}+\cdots+\mu_nx^{2n}a^{-2n}}$$

are considered. This paper connects with the previous work of the author and of N. I. Muschelišvili on integral equations with integrals in the sense of Cauchy principal values.

W. J. Trjitzinsky (Urbana, Ill.).

Calculus of Variations

Bieri, Hans. Ein Beispiel zum Randwertproblem der Variationsrechnung. Comment. Math. Helv. 17, 262–277 (1945).

The paper is concerned primarily with the extremals of integrals of the form

$$\int\{a(x\dot{y}-y\dot{x})+g(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}\}dt,$$

where $g^3=4a^3x^3+c$. These extremals are expressible as polynomials in t of degree 3. The author discusses in detail the envelope of the extremals through a point. This problem is one of the few in which the results can be expressed in terms of elementary functions. *M. R. Hestenes.*

Flodin, Bertil. Über diskontinuierliche Lösungen bei Variationsproblemen mit Gefällbeschränkung. Acta Soc. Sci. Fennicae. Nova Ser. A. **3**, no. 10, 31 pp. (1945). [MF 14655]

This paper considers the problem of minimizing an integral

$$I=\int_{x_1}^{x_2} f(x, y, y')dx,$$

where admissible curves lie in the (x, y)-plane and their slopes are subject to the restrictions $a\leqq y'\leqq b$. Both necessary and sufficient conditions are developed in considerable detail for the case when the minimizing curve has a single corner and one of its component arcs is a straight line segment with slope equal to either a or b, while the other arc is either a similar segment or an arc of an extremal. [A much more general problem was treated by F. A. Valentine [Contributions to the Calculus of Variations, University of Chicago Press, 1937, pp. 403–447], but, owing to restrictive assumptions, Valentine's paper does not include the results of the present memoir.] *L. M. Graves.*

Pauc, Christian. L'intégrale de Weierstrass sur les courbes non rectifiables. Rev. Sci. (Rev. Rose Illus.) **78**, 299–302 (1940). [MF 13320]

Let H be a region of n-space E_n and let $\varphi(x, x')$ be defined for x in H and x' in E_n. An integrand function φ on (H, E_n) is positively homogeneous of degree 1 in x'. If C is a curve in E_n, the author defines the integral of Weierstrass of φ along C to be the length $\lambda\varphi(C)$ of C in a metric space $E_{n\varphi}$ of points in E_n. If $x=(\xi_1, \cdots, \xi_n)$, $y=(\eta_1, \cdots, \eta_n)$ are two points in E_n, the distance between them in $E_{n\varphi}$ is given by $\varphi(\xi_1, \cdots, \xi_n, \eta_1-\xi_1, \cdots, \eta_n-\xi_n)$. If C is rectifiable and φ is continuous, $\lambda\varphi(C)$ exists, is finite and is equal to the integral of Lebesgue and Tonelli. In 1936 Menger proposed the following question: if C is nonrectifiable does there exist a continuous and quasi-regular (convex in x') integrand φ for which the upper and lower integrals of Weierstrass of φ along C are unequal? The author answers the question affirmatively. Moreover, he constructs for every C such that $\lambda(C)=+\infty$ an integrand φ, linear in x' with coefficients that are continuous in x, for which the upper and lower integrals of Weierstrass are unequal. *H. H. Goldstine.*

Debever, Robert. Quelques conséquences de la condition nécessaire d'Hadamard du calcul des variations des intégrales multiples. Bull. Soc. Roy. Sci. Liége **10**, 584–589 (1941). [MF 13085]

The author considers multiple integral problems of the form

$$I=\int_D L(t^\alpha, x^i, x^i_\alpha)dt^1\cdots dt^\mu=\text{minimum},$$
$$\alpha=1, \cdots, \mu;\ i=1, \cdots, n,$$

where, in evaluating I for a given set V of functions $x^i=x^i(t^1, \cdots, t^\mu)$, x^i_α is to be replaced by $\partial x^i/\partial t^\alpha$; D is assumed convex. If such a set V is of class C'' and furnishes a weak relative minimum for I among nearby sets with the same boundary values, then Hadamard's necessary condition states that at each element Σ_0 : $(t_0^\alpha, x_0^i, x^i_{\alpha 0})$ of V, the biquadratic form

$$L_{\alpha\beta}{}^{ij}_{\gamma\delta}\lambda^\alpha\lambda^\beta\nu_i\nu_j\geqq 0$$

for all values of the λ and ν. An element Σ_0 is said to be positive definite if the form above takes on positive values except when either all the λ or all the ν (or both) are zero.

The author proves two theorems. (1) If Σ_0 is a positive definite element, there exist constants $A^{\alpha\beta}_{ij}$ with (A) $A^{\alpha\beta}_{ij}=-A^{\beta\alpha}_{ij}=-A^{\alpha\beta}_{ji}=A^{\beta\alpha}_{ji}$ such that the determinant (of order $n\mu$)

$$D=|L_{\alpha\beta}{}^{ij}_{\gamma\delta}-A^{\alpha\beta}_{ij}|\neq 0.$$

(2) If L is also analytic, it is possible in a great many ways (specified in more detail in the paper) to find an n-parameter family of extremal surfaces, at least one of which passes through Σ_0. In particular, a sufficiently small portion of any given analytic extremal surface through Σ_0 can be embedded in such a family.

The author uses the first theorem to show that a weak neighborhood of an extremal surface V near a positive definite element Σ_0 can be incorporated in a "geodesic field" as defined by H. Weyl [Ann. of Math. (2) **36**, 607–629 (1935)] for the problem defined by $L^*\equiv L-A^{\alpha\beta}_{ij}x^i_\alpha x^j_\beta$, which is equivalent to L by virtue of relations (A) (for fixed boundary values). The author has shown in a previous paper [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **23**, 809–815 (1937)] that the results of theorem (2) imply the existence of a field of Mayer which permits the construction of an "excess function" and the classical transformation of the second variation which was performed by Clebsch.

C. B. Morrey, Jr. (Berkeley, Calif.).

Theory of Probability

Kaplansky, Irving. The asymptotic distribution of runs of consecutive elements. Ann. Math. Statistics **16**, 200–203 (1945). [MF 12911]

For a permutation of $1, 2, \cdots, n$, let r denote the number of times in which two consecutive numbers occur next to each other. For example, in 234651, $r=3$. Assuming that all permutations are equally probable, and denoting the probability of r by $P(n, r)$ and the kth factorial moment of r by $M(n, k)$, the author shows that

$$P(n, r)=\frac{2^r e^{-2}}{r!}\left\{1-\frac{r^2-3r}{2n}+\frac{r^4-8r^3+9r^2+22r-16}{8n(n-1)}\right\}+O(n^{-3}),$$

$$M(n, k)=2^k\left\{1-\frac{k+1}{2k}\binom{k}{1}\frac{k}{n}+\frac{k+2}{2^2k}\binom{k}{2}\frac{k(k-1)}{n(n-1)}-\cdots\right\}.$$

From either of these expressions follows the result of Wolfowitz [same Ann. **15**, 97–98 (1944); these Rev. **6**, 5] that r has asymptotically the Poisson distribution with expectation 2. The expression for $P(n, r)$ shows that the approximation by this Poisson distribution is more rapid for $r=0$ and $r=3$ than for other values.

Z. W. Birnbaum (Seattle, Wash.).

Smith, Jerome C. Asymptotic distribution of sums of Rademacher functions. Bull. Amer. Math. Soc. **51**, 941–944 (1945). [MF 14465]

In the case of independent random variables taking on the values ± 1, each with probability $\frac{1}{2}$, the central limit theorem may be stated as follows. Let $r_n(t)=\text{sign sin } 2^n\pi t$

be the Rademacher functions. Then the measure of the set of the values t $(0 \leqq t \leqq 1)$ such that

$$(*) \qquad (2n)^{-\frac{1}{2}} \left| \sum_{k=1}^{n} r_k(t) \right| < a$$

tends to $\pi^{-\frac{1}{2}} \int_{-a}^{a} \exp(-u^2) du$ as $n \to \infty$. The author shows, by using the continuity theorem for Fourier-Stieltjes transforms, that if in $(*)$ we replace a by a measurable function $f(t)$, the corresponding measure tends to

$$\pi^{-\frac{1}{2}} \int_0^1 \int_{-f(t)}^{f(t)} e^{-u^2} du$$

[this result can also be deduced directly from the preceding one by approximating $f(t)$ by step functions with discontinuities at binary rationals]. A formula is also obtained for the limit measure of the set of t's satisfying $(*)$, where n is replaced by a sequence of integral-valued functions $N_n(t)$. [The continuity theorem for the Fourier-Stieltjes transform is not stated quite clearly by the author. In particular, the assumption $f(t) = \lim f_n(t)$ on p. 941 should have been omitted.] *A. Zygmund* (Philadelphia, Pa.).

Bottema, O., and van Veen, S. C. Calculation of probabilities in the game of billiards. I. Nieuw Arch. Wiskde. (2) **22**, 15–33 (1943). (Dutch) [MF 14302]

Typical for the problems treated are the following examples (translated into the customary terminology of probability theory). In a series of Bernoulli trials one requires the probability distribution of the number of successes in the first r runs of successes, the probability that the Nth success occurs in the ρth run, and the expected value of ρ; in two independent Bernoulli sequences, one requires the probability that the Nth successes occur in the same run. The last probability is obtained in finite form using properties of the hypergeometric series.

W. Feller (Ithaca, N. Y.).

Täcklind, Sven. Sur le risque de ruine dans des jeux inéquitables. Skand. Aktuarietidskr. **25**, 1–42 (1942). [MF 14135]

In the first part the integral equation

$$(1) \qquad \varphi(x) = F(-x) + \int_0^\infty \varphi(y) dF(y-x)$$

is studied, where $F(x)$ is a distribution function with positive mean value and such that the integral of $\exp(-ay) dF(y)$ converges for some $a > 0$. A solution is derived by means of successive approximations using essentially the Wiener-Hopf method. The results are applied to the following probability problem occurring in the F. Lundberg-Cramér theory of risk for insurance companies and considered by other methods by C.-O. Segerdahl [On Homogeneous Random Processes and Collective Risk Theory, Uppsala, 1939]. Let a player A (the company) have an initial capital x and let it play against a player B with an unlimited capital. A sequence of games takes place at moments $t_1 < \cdots < t_n < \cdots \to \infty$. Let $\pi(y)$ be a distribution function with finite mean value M and denote by $\pi^{k*}(y)$ its k-fold iterated convolution with itself. The probability that at the nth game A wins an amount not exceeding y is supposed to be $F(y, t_n - t_{n-1})$ with

$$(2) \qquad F(y, t) = e^{-\lambda t} \sum_{k=0}^{\infty} t^k \pi^{k*}(y - (\lambda - M)t)/k!,$$

where λ is a constant. The probability $\varphi(x, \{t_r\})$ of the ruin of A is desired. If, in particular, $t_n = nt$, this function reduces to a solution $\varphi(x, t)$ of

$$(3) \qquad \varphi(x, t) = F(-x, t) + \int_0^\infty \varphi(y, t) d_y F(y - x, t).$$

Now $\varphi(x, t) \to \varphi(x, 0)$ as $t \to 0$, and $\varphi(x, 0)$ is a solution of an equation of type (1). The practical importance of $\varphi(x, 0)$ depends on the fact, proved by the author, that $\varphi(x, \{t_r\}) \leqq \varphi(x, 0)$. Thus effective estimates for the desired probability of ruin are obtained. *W. Feller.*

Schärf, Henryk. Über partielle Bestandsänderungen und eine Klasse neuer Integrationsprozesse. Mitt. Verein. Schweiz. Versich.-Math. **44**, 233–249 (1944). [MF 13963]

Denote by $L(t)$ the number of elements in a population at time t, $a \leqq t \leqq b$, and by $f_i(t)$, $i = 1, \cdots, n$, the number leaving by the ith exit in the time interval (a, t), so that $L(t) = L(a) - \sum f_i(t)$. Let

$$p_i(J) = p_i(1/L(\alpha); \Delta f_1(\alpha), \cdots, \Delta f_n(\alpha)) = p_i(y; z_1, \cdots, z_n)$$

denote the influence of the ith exit on $L(t)$ in the time interval $J = (\alpha, \beta)$, that is, $p_i(y; 0, \cdots, 0) = 1$;

$$\partial p_i(y; 0, \cdots, 0)/\partial z_j = -\delta_{ij} y.$$

Extending a result of Loewy [Bl. Versich. Math. **2**, 207–216 (1932)], the author shows that

$$\prod_{s=0}^{n-1} p_i(J_s) \to \Pi_i = \exp\left(-\int_a^b \{L(t)\}^{-1} df_i(t) \right)$$

as $|J_s| \to 0$, where $\{J_s\}$ is a subdivision of (a, b), so that $L(b)/L(a) = \Pi_1 \Pi_2 \cdots \Pi_n$. *D. Blackwell.*

Wald, Abraham. Some generalizations of the theory of cumulative sums of random variables. Ann. Math. Statistics **16**, 287–293 (1945). [MF 13919]

Let $\{z_i\}$ be independent chance variables with identical distributions; let $k_i = k_i(z_1, \cdots, z_i)$, where $P\{|k_i| \geqq 1 \text{ for some } i\} = 1$. Let n be the smallest integer for which $|k_i| \geqq 1$; define $w = z_1 + \cdots + z_n$. It is shown that $E(w) = E(z)E(n)$ if (a) $P\{n < N\} = 1$ for some N or (b) $E(n)$ and the first four moments of z are finite. Lower bounds are obtained for $E(n)$ which imply the efficiency of the sequential probability ratio test. The results are extensions of earlier results in a special case (essentially $k_i = z_1 + \cdots + z_i$) [same Ann. **15**, 283–296 (1944); these Rev. **6**, 88]. *D. Blackwell.*

Roy, Samarendra Nath, and Bose, Purnendu. Bernoulli's theorem and Tshebycheff's analogue. Sankhyā **7**, 209–210 (1945). [MF 15137]

Loève, Michel. Nouvelles classes de lois limites. Bull. Soc. Math. France **73**, 107–126 (1945). [MF 14178]

Details of proofs of theorems announced in C. R. Acad. Sci. Paris **210**, 202–204 (1940); these Rev. **1**, 246, and some further developments on distribution functions $F(x)$ which are divisible by the distribution function $F(x/\alpha)$, where α is a constant less than 1. *J. L. Doob* (Urbana, Ill.).

***Arley, Niels. On the Theory of Stochastic Processes and their Application to the Theory of Cosmic Radiation.** Thesis, University of Copenhagen, 1943. 240 pp. (English. Danish summary)

In the first part (pp. 19–84) and an appendix (pp. 193–221) the author surveys the theory of Markov processes

with denumerably many possible states and develops existence theorems for the infinite systems of linear differential equations which occur. This theory has later been generalized and pushed further by the author and V. Borchsenius [Acta Math. 76, 261–322 (1945); these Rev. 7, 161]. Special consideration is given to the case where the states are ordered according to several subscripts and the transition probabilities accordingly form multidimensional matrices. In the second part (pp. 87–190) the author constructs special models suitable for the theory of cosmic radiation. Also, the older literature is surveyed from the viewpoint of stochastic processes. This part contains a lengthy discussion of the results from a physical point of view and a comparison with experiments.

A second appendix (pp. 222–227) is devoted to a discussion of numerical evaluations of $\psi(x)=\int_0^x \exp(t^2)dt$. Tables to four significant figures are given for values $2\leqq x\leqq 10$ in steps of .01. Similar tables are given for $2x\psi(x)\exp(-x^2)$.

W. Feller (Ithaca, N. Y.).

Arley, Niels. On the elementary, time-homogeneous, discontinuous, stochastically definite process. Skand. Aktuarietidskr. 27, 172–176 (1944). [MF 14149]

The differential equations of radioactive disintegration, $x_n'=-p_nx_n+p_{n-1}x_{n-1}$, are solved by means of a discontinuous factor. [The Laplace transform yields the same result more directly. For another derivation by the same author, cf. Mat. Tidsskr. B. 1939, 49–51 (1939); these Rev. 1, 72.]

W. Feller (Ithaca, N. Y.).

Doob, J. L. Markoff chains—denumerable case. Trans. Amer. Math. Soc. 58, 455–473 (1945). [MF 14232]

Let $p_{ij}(t)$ denote the transition probabilities of a stationary Markov chain, $i, j\geqq 1$. Analytically, the $p_{ij}(t)$ may be any continuous nonnegative functions satisfying the Kolmogoroff equation $\sum_j p_{ij}(s)p_{jk}(t)=p_{ik}(s+t)$, the initial condition $p_{ij}(0)=\delta_{ij}$ and

$$\sum_j p_{ij}(t)=1. \tag{1}$$

It is known that under weak regularity conditions the $p_{ij}(t)$ satisfy the two systems of differential equations

$$p_{ij}'(t)=-q_ip_{ij}(t)+\sum_k q_{ik}p_{kj}(t), \tag{2}$$

$$p_{ij}'(t)=-q_jp_{ij}(t)+\sum_k q_{kj}p_{ik}(t), \tag{3}$$

where $t>0$ and $q_i=-p_{ii}'(0)$, $q_{ij}=p_{ij}'(0)$. By successive approximations one can construct a solution of these equations satisfying all conditions except perhaps (1), which has to be replaced by (1′) $p_{ij}(t)<1$ [Feller, same Trans. 48, 488–515 (1940); these Rev. 2, 101]. The present paper combines the study of the above equations with a direct study of the sample function $x(t)$ representing the process. It turns out, in particular, that in the case where the q_i are finite and (1) is satisfied the truth of (2) or (3), respectively, is equivalent to the fact that the first discontinuity of $x(t)$ after or before any specified t_0 is a jump. More precise information is obtained in the case where the discontinuities of $x(t)$ are well-ordered. For given q_i, q_{ij} there are only two possibilities. (a) The equations (2) and (3) have a unique solution satisfying (1); in that case there is probability one that in any finite time interval there are only finitely many transitions, and all singularities are jumps (necessary and sufficient conditions for that case were given by Feller, loc. cit.). (b) The probability that all singularities are jumps is less than 1. Then (2) is satisfied, but in (3) the equality has to be replaced by the sign $\geqq$. In that case there exist infinitely many Markov processes with the same q_i, q_{ij}, all of which satisfy (2); furthermore, for all these processes the probabilities of transition in finitely many steps are the same and satisfy both (2) and (3).

W. Feller (Ithaca, N. Y.).

Blackwell, David. Finite non-homogeneous chains. Ann. of Math. (2) 46, 594–599 (1945). [MF 14123]

The author considers Markov chains depending on a finite number of states. These are determined by matrix functions $P(s,t)$ satisfying the equation $P(r,s)P(s,t)=P(r,t)$ for $r<s<t$, where the parameters r, s, t are integral valued. The matrix elements are nonnegative and all row sums are 1. (The (i,j)th element of $P(s,t)$ is the probability of a transition to state j at time t from a state i at time s.) Continuing the work of Kolmogoroff [Math. Ann. 112, 155–160 (1935)], the author investigates the limiting values of $P(s,t)$ as $s\to-\infty$ and analyzes their significance, finding the exact analogues of the ergodic sets in state space whose properties are classical in the stationary case, when $P(r,s)=P(0,1)^{s-r}$. In particular, as in the stationary case, he finds that, if there is a positive ϵ with the property that for some $t>0$ all the elements of $P(r,r+t)$ are not less than ϵ for infinitely many negative values of r, then $\lim_{r\to-\infty}P(r,s)$ exists, and is a matrix with identical rows.

J. L. Doob.

Opatowski, I. Markoff chains with reverse transitions. Proc. Nat. Acad. Sci. U. S. A. 31, 411–414 (1945). [MF 14531]

Consider the system of $n+1$ ordinary differential equations

$$\begin{aligned} Y_0' &= -(k+g)Y_0+gY_1, \\ Y_k' &= -(k+g)Y_k+gY_{k+1}+kY_{k-1}, \qquad 1\leqq k\leqq n-1, \\ Y_n' &= -aY_n+kY_{n-1}, \end{aligned}$$

where k and g are positive constants and a stands for either g, $k+g$ or zero. The initial conditions are $Y_0(0)=1$, $Y_k(0)=0$ for $1\leqq k\leqq n$. Using Laplace transforms the author determines Y_n (while the remaining Y_k are said to be of minor practical importance).

W. Feller (Ithaca, N. Y.).

Barkalaia, A. Sur des chaines de Markoff. Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 33–36 (1944). (Russian. French summary) [MF 15190]

Let $\{p_{ij}\}$, $i, j\leqq n$, be the transition probabilities of a Markov chain. Let $m_{ij}(N)$ be the number of times the system, starting from state i, passes through state j, in taking N steps. It is proved that (assuming any initial state i) $\lim_{N\to\infty}N^{-1}m_{ij}(N)$ exists, with probability 1. The proof is elementary. Kolmogoroff has pointed out that the theorem can also be deduced from the ergodic theorem. [See Fréchet, Méthode des Fonctions Arbitraires. Théorie des Événements en Chaine dans le Cas d'un Nombre Fini d'États Possibles, Gauthier-Villars, Paris, 1938, pp. 159–160.]

J. L. Doob (Urbana, Ill.).

Dubrovsky, V. M. On a problem connected with purely discontinuous random processes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 459–461 (1945). [MF 14421]

The author considers a Markov process where the possible states form an abstract space $\mathfrak{R}$ and where only a finite number of transitions occur in any finite time interval. Let $s<t$ be two values of the time parameter and let $\mathfrak{A}$ and $\mathfrak{B}$ be two measurable subsets of $\mathfrak{R}$. For any fixed $\lambda<t-s$

let $\pi_n(s, x, t, \mathfrak{A}, \lambda, \mathfrak{B})$ denote the conditional probability of the simultaneous realization of the following three events evaluated on the hypothesis that at time s the state is given by the point x: (1) during (s, t) exactly n events occur; (2) at time t the state is a point of $\mathfrak{A}$; (3) during (s, t) the state is in the set $\mathfrak{B}$ at a set of moments whose Lebesgue measure exceeds λ. It is shown how the π_n can be determined by integral recurrence formulas from the infinitesimal transition probabilities (which, of course, have to be known). It is also shown that an estimate of the type $\pi_n < c^n(t-s)^n/n!$ is generally valid. *W. Feller.*

Blanc-Lapierre, André. Sur l'effet de scintillation. C. R. Acad. Sci. Paris **221**, 375–377 (1945). [MF 14678]

The author investigates the stochastic process determined as follows: particles are arriving on a surface at a uniform rate ρ; the probability that a particle which has arrived at time t_0 leaves between t and $t+dt$ time units later is $e^{-t/\alpha}dt/\alpha$. Let $n(t)$ be the number of particles on the surface at time t. The characteristic function of the joint distribution of $n(t_1)$, $n(t_2)$ is calculated. When $\rho\tau \to \infty$ the $n(t)$ stochastic process is asymptotically the stationary Gaussian Markov process, with correlation function a constant multiple of $e^{-t/\alpha}$. Bernamont [same C. R. **198**, 1755–1758 (1934)] investigated this same process and found the spectrum of the limiting Gaussian process. This process is important in the theory of spontaneous thermal current fluctuations.

J. L. Doob (Urbana, Ill.).

Bass, Jean, Dedebant, Georges, et Wehrlé, Philippe. Les équations différentielles aléatoires. C. R. Acad. Sci. Paris **221**, 168–171 (1945). [MF 14247]

The authors consider differential equations of the form (1) $X_i' = F_i(X_1, \cdots, X_n, t)$, $i \leqq n$, where the X_i are chance variables. The stochastic element is derived from the fact that the constants of integration are chance variables. Using results obtained by Dedebant, Moyal and Wehrlé [same C. R. **210**, 243–245 (1940); these Rev. **1**, 246] they find that the probability density of distribution of $X_1(t)$, $\cdots$, $X_n(t)$ satisfies the equation

$$\partial\rho/\partial t + \sum_i \partial(\rho F_i)/\partial x_i = 0.$$

In particular, they discuss the stationary isotropic solutions of the system $X''+\omega^2 X=0$, $Y''+\omega^2 Y=0$ (which is easily reduced to the form (1) with $n=4$), finding

$$\rho = \Phi\{\omega^2(X^2+Y^2)+X'^2+Y'^2,\ Y'X - X'Y\}.$$

This corresponds [Dedebant, Moyal and Wehrlé, same C. R. **210**, 332–334 (1940)] to a hydrodynamical rotation flow whose angular velocity varies with the distance from the center, which reduces to a rigid rotation if Φ is Gaussian.

J. L. Doob (Urbana, Ill.).

Mathematical Statistics

Burrau, Øjvind. On the determination of the mean error. Mat. Tidsskr. B. **1945**, 97–109 (1945). (Danish) [MF 14264]

Continuation, in greater detail, of the critique of the (classical) theory of the mean error begun in Mat. Tidsskr. B **1943**, 9–16 (1943); these Rev. **7**, 130. *W. Feller.*

Fréchet, Maurice. Sur une loi de probabilité considérée par J. F. Steffensen. Skand. Aktuarietidskr. **24**, 214–220 (1941). [MF 14134]

Cf. Skand. Aktuarietidskr. **24**, 1–12 (1941); these Rev. **3**, 5.

Reiersöl, Olav. Measures of departure from symmetry. Skand. Aktuarietidskr. **27**, 229–234 (1944). [MF 14150]

The author points out that, for most measures which have been proposed for the skewness of a frequency distribution, a zero value is a necessary but not a sufficient condition for symmetry. He remarks that a necessary and sufficient condition for the set of values $x_1, \cdots, x_n$ of the variable x arranged in order of magnitude to be symmetrical is that $x_i + x_{n-i+1}$ is constant ($i=1, \cdots, n$). Regarding x_i as a value of x and x_{n-i+1} as a value of y ($i=1, \cdots, n$), it is proposed that departure from symmetry be measured by the ratio of a measure of the dispersion of $x+y$ to the measure of the dispersion of x. The measure of skewness actually considered is that in which the measure of dispersion is the standard deviation and the ratio is multiplied by the factor $1/\sqrt{2}$. Such a measure is obviously independent of the origin and the scale and it is shown that its value lies between 0 and 1. *C. C. Craig* (Ann Arbor, Mich.).

Ottestad, Per. On Bernoullian, Lexis, Poisson and Poisson-Lexis series. Skand. Aktuarietidskr. **26**, 15–67 (1943). [MF 14141]

Let X_{ij}, $j=1, \cdots, n_i$; $i=1, \cdots, v$, be independent variables assuming values 0 and 1; $P\{X_{ij}=1\} = c_i+q$ or c_iq; $x = \sum_{i,j} X_{ij}$, $n = \sum n_i$, where n and q are considered as independent random variables. Various formulas are obtained, including those for moments of x, regressions of x on q and n, and correlations between x and q and x and n, which specialize to known results for Poisson and Lexis series.

D. Blackwell (Washington, D. C.).

Ottestad, Per. On certain compound frequency distributions. Skand. Aktuarietidskr. **27**, 32–42 (1944). [MF 14146]

Let x have a distribution $F(x; y, z, \cdots)$, where $y, z, \cdots$ are parameters. If the parameters are themselves chance variables, the distribution $F(x)$ of x is a function of the distributions of the parameters. Three examples are given.

D. Blackwell (Washington, D. C.).

Gaddum, J. H. Lognormal distributions. Nature **156**, 463–466 (1945). [MF 14013]

A survey of various papers treating the logarithmic transformation connected with a Gaussian random variable. The author appears to suggest that a logarithmic scale should be used whenever "scientific observations show uncontrolled variations large compared with the observations themselves." *W. Feller* (Ithaca, N. Y.).

Quensel, Carl-Erik. Studies of the logarithmic normal curve. Skand. Aktuarietidskr. **28**, 141–153 (1945). [MF 14430]

For the logarithmic normal distribution higher moments are determined in terms of the first three. A criterion of a successful fit of this function to data is that the fourth moment computed from the data does not differ significantly from that for the graduating function. The author gives a large-sample expression for the standard error of a criterion of this type. He next points out that, if the lower limit of a

logarithmic normal distribution is assumed known, such a function can be fitted to data by the use of the first two moments, and he gives a large-sample standard error for a third-moment criterion to be used in this case. Next he considers the logarithmic type A series function, with the lower limit known, and again develops large-sample standard errors for criteria that the semi-invariants (of Thiele) of order greater than n ($n=3, 4, 5, 6$) can be neglected. Finally, he develops a method for fitting a logarithmic normal function to a truncated observed distribution.

C. C. Craig (Ann Arbor, Mich.).

Höffding, Wassilij. Stochastische Abhängigkeit und funktionaler Zusammenhang. Skand. Aktuarietidskr. **25**, 200–227 (1942). [MF 14138]

Let X, Y be chance variables with individual continuous distribution functions $F_1(x)$, $F_2(y)$ and joint distribution function $F(x, y)$. The author defines "C-dependence" as the existence of an (x, y)-set C with $\iint_C dF(x, y)=1$, $\iint_C dF_1(x)dF_2(y)=0$, and shows that this is a very strong type of dependence. He shows that

$$\gamma=\max_{\mathfrak{M}}\iint_{\mathfrak{M}} dF(x, y)-\iint_{\mathfrak{M}} dF_1(x)dF_2(y)$$

is a measure of dependence which satisfies the conditions that are normally desired of such a measure: $\gamma=0$ for independence, $\gamma=1$ for C-dependence and otherwise $0<\gamma<1$. The measure γ is related to Steffensen's ω [Skand. Aktuarietidskr. **24**, 13–33 (1941); these Rev. **3**, 5] by the equation $\omega=2\gamma/(1+\gamma)$. It is shown that Pearson's contingency coefficient may be arbitrarily near 1 at the same time that γ is arbitrarily near 0. This fact is interpreted as a serious criticism of the contingency coefficient. Finally, the author discusses the proper applications to continuous distributions approximated by discrete distributions by means of class division, and corresponding measures of dependence for discrete distributions. *J. L. Doob* (Urbana, Ill.).

Lakshmanamurti, M. Coefficient of association between two attributes in statistics. Proc. Indian Acad. Sci., Sect. A. **22**, 123–133 (1945). [MF 14452]

Féraud, Lucien. Paramètre ignorable dans une loi de probabilité. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 58–61 (1945). [MF 14558]

Féraud, Lucien. Statistique mathématique: Distributions de produits intérieurs. C. R. Séances Soc. Phys. Hist. Nat. Genève **60**, 196–200, 296 (1943). [MF 14218]

A derivation of the well-known distribution functions of the second-order product moment both about zero and about the means in samples from a normal bivariate universe, starting from the equally well-known fact that these quantities may be written as the difference of two independently distributed sums of squares of normally distributed variates. *C. C. Craig* (Ann Arbor, Mich.).

Féraud, Lucien. Critères statistiques applicables à un petit nombre d'observations. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 116–118 (1942). [MF 14202]

Assume t_1 and t_2 independently distributed according to t-distributions with n_1 and n_2 degrees of freedom, respectively, and let $q_i=t_i^2/n_i$. The distributions of q_1q_2 and of $\{(1+q_1)(1+q_2)\}^{-1}$ are found in terms of the hypergeometric function. *H. Scheffé* (Princeton, N. J.).

Krishna Iyer, P. V. The use of generalised Dirichlet's integral in solving some distribution problems of statistics. Proc. Indian Acad. Sci., Sect. A. **22**, 75–83 (1945). [MF 14453]

Alternative derivations of some well-known statistical distributions. *J. Wolfowitz* (Raleigh, N. C.).

Quensel, Carl-Erik. An extension of the validity of "Student"-Fisher's law of distribution. Skand. Aktuarietidskr. **26**, 210–219 (1943). [MF 14143]

Samples $(x_1, \cdots, x_N)$ are considered in which the x_i are independently normally distributed, each with mean zero but with variances not assumed equal. With the aid of moment-generating functions, approximations are obtained for the joint and marginal distributions of the first and second sample moments and for the distribution of the Student ratio formed for the sample. The relation of the last approximation to the customary t-distribution is discussed. *H. Scheffé* (Princeton, N. J.).

Simonsen, W. On distributions of functions of samples from a normally distributed infinite population. Skand. Aktuarietidskr. **27**, 235–261 (1944). [MF 14151]

Simonsen, W. On distributions of functions of samples from a normally distributed infinite population. II. Skand. Aktuarietidskr. **28**, 20–43 (1945). [MF 14153]

An exposition of known results on the distributions of moments, correlation coefficients, etc. *J. Wolfowitz.*

Gårding, Lars. The distributions of the first and second order moments, the partial correlation coefficients and the multiple correlation coefficient in samples from a normal multivariate population. Skand. Aktuarietidskr. **24**, 185–202 (1941). [MF 14132]

This is a well organized exposition, both concise and clear, of the derivation of the distributions named in the title.

C. C. Craig (Ann Arbor, Mich.).

Janardana Aiyer, S. On the arithmetic and the geometric means from a type III population. Math. Student **13**, 11–15 (1945). [MF 14489]

Shafei, A. M. N. On the standard deviation of samples drawn from a type III distribution. Proc. Math. Phys. Soc. Egypt **1**, no. 3, 1–8 (1939). [MF 14104]

The author develops the probability density dP that a sample has a standard deviation between σ and $\sigma+d\sigma$ for samples of 2 and 3 taken from a population of the Pearson type III, $y=(1/\Gamma(p))x^{p-1}e^{-x}$. For a sample of 2, $dP=k(2\sigma)^m K_m(2\sigma)d\sigma$, where $m=p-\frac{1}{2}$ and k is a constant. For a sample of 3, dP is much more complicated, being composed of a series of integrals of the form

$$I(a)=\int_0^{\pi/3} e^{-a\cos\theta}\sin^{2n}\theta d\theta.$$

The general solution of I (a) for all n is given.

R. L. Anderson (Raleigh, N. C.).

Hsu, P. L. On the power functions of the E^2-test and the T^2-test. Ann. Math. Statistics **16**, 278–286 (1945). [MF 13918]

The author has previously shown that any linear hypothesis concerning one or several p-variate normal universes having common variances and covariances can be reduced to a canonical form in which the sampling distribution is

an $(m+n)p$-variate normal density function of the variables y_{ir} $(i=1, \cdots, f;\ r=1, \cdots, m)$ having means η_{ir} and an independently distributed set Z_{is} $(s=1, \cdots, n)$ with zero means [Ann. Eugenics 11, 42–46 (1941); these Rev. 3, 8]. The common p-dimensional variance-covariance matrix is unknown and the hypothesis H to be tested is that $\eta_{ir}=0$ $(i=1, \cdots, f;\ r=1, \cdots, n_1;\ n_1 \leqq m)$. The present paper explores further the power functions of the tests E^2 and T^2 which have been arrived at in the two cases: (1) $p=1$, that is, $m+n$ observations are taken from $m+n$ normal univariate universes with a common variance σ^2 in which the means of n are known to be zero and it is hypothecated that the means of n_1 of the remaining m are also zero; (2) $n_1=1$, that is, the hypothesis is that in one of the m p-variate universes of means η_{ir} sampled the means were all zero. In case (1), H becomes $H' : \eta_r=0$ $(r=1, \cdots, n_1)$ and it is known that the power function of the critical region w_0 for the test E^2 depends only on $\lambda=\sum_{i=1}^{n_1}\eta_i^2/\sigma^2$. Wald [Ann. Math. Statistics 13, 434–439 (1942); these Rev. 5, 129] had generalized the author's earlier theorem that w_0 is uniformly most powerful of the regions whose power functions depend on λ alone to establish a maximum property of w_0 with respect to the integral of the power function over the surface on which λ is constant. Hsu here gives an easier proof of Wald's theorem and further generalizes the result to show the similar maximum property within a volume. In case (2) analogous theorems for T^2 are discussed. The theorem given analogous to the last result in case (1) is less satisfactory due to the fact that the integral involved is not always finite and it proves not to include Simaika's result [Biometrika 32, 70–80 (1941); these Rev. 2, 236] analogous to the first of the three of case (1). A modification to introduce a finite integral yields the interesting result that other tests than T^2 exist with desirable maximum properties.

C. C. Craig (Ann Arbor, Mich.).

Radhakrishna Rao, C. Studentised tests of linear hypotheses. Science and Culture 11, 202–203 (1945). [MF 14551]

This paper discusses five problems involving normal distributions whose covariance matrices are unknown, and linear hypotheses about these distributions. The author is evidently unaware that the tests described by him are not new. For example, problem 1, to test the equality of the means of a p-variate normal population, is solvable by Hotelling's generalized T; the author's test is the same.

J. Wolfowitz (Raleigh, N. C.).

Stein, Charles. A two-sample test for a linear hypothesis whose power is independent of the variance. Ann. Math. Statistics 16, 243–258 (1945). [MF 13915]

Denote by $x_1, x_2, \cdots$, independent observations from a normal population with unknown mean ξ and unknown variance σ^2, and consider tests of the hypothesis $H : \xi=\xi_0$, and confidence intervals for ξ. It was shown by Dantzig [same Ann. 11, 186–192 (1940); these Rev. 1, 348] that for fixed n there exists no test of H based on a sample $x_1, \cdots, x_n$, whose power is independent of σ^2 (with trivial exceptions where the power is constant). Hence there is great theoretical interest in the following result obtained in this paper. There exist two-sample tests of H, the first sample being of fixed size n_0 and the second sample having a (random) size $n-n_0$ determined by the first, whose power is independent of σ^2. The power of the test may be calculated from the t-distribution instead of the noncentral t-distribution. In terms of confidence intervals the result may be stated more completely as follows. Choose a length l for the confidence interval, a confidence coefficient $1-\alpha$ and a positive integer n_0. Take a first sample of n_0. From this the second sample size $n-n_0$ is determined, and from the total sample of n a confidence interval with the specified properties may be constructed. Bounds are found for the expected sample size $E(n)$. For practical applications a modification of these tests and confidence intervals is proposed with the result that the tests have greater power for $\xi \neq \xi_0$, the confidence intervals have a confidence coefficient greater than $1-\alpha$, and $E(n)$ is slightly less than before. The modified confidence intervals may be described thus: from the first sample $x_1, \cdots, x_{n_0}$ calculate $s^2=\sum_1^{n_0}(x_i-\bar{x}')^2/(n_0-1)$, where $\bar{x}'=\sum_1^{n_0}x_i/n_0$. Define $z=(\frac{1}{2}l/t_0)^2$, where t_0 is the upper $100\alpha/2$ percent point on the t-distribution with n_0-1 degrees of freedom. Define n as the larger of n_0 and the smallest integer not less than s^2/z. Take a second sample of size $n-n_0$, denoted by $x_{n_0+1}, \cdots, x_n$. Let $\bar{x}=\sum_1^n x_i/n$. The desired confidence interval is $(\bar{x}-\frac{1}{2}l, \bar{x}+\frac{1}{2}l)$. It is shown that under some conditions $E(n)$ compares favorably with the fixed sample size that would be required for the usual single-sample confidence intervals to get the same l if σ^2 were known. The question as to a good choice of n_0 is not raised.

These results are extended to tests and confidence regions for a "linear hypothesis" (this includes analysis of variance and covariance) under the usual assumptions of normal regression theory, the design being replicated a number of times depending on the outcome of the first sample. The power of the tests involves a new distribution function called the "non-central F-distribution," namely, that of $F'=\sum_1^m(q_i-c_i)^2$, where $q_i=x_i/r^{\frac{1}{2}}$, the x_i are independent standard normal deviates, r is distributed independently of the x_i like χ^2 with n degrees of freedom, and the c_i are constants. The probability density function of F' is given in the form of a definite integral, so that $\Pr\{F' < F_0'\}$ is left as a double integral. The result of Dantzig mentioned above is generalized to the case of tests of linear hypotheses.

[In view of all these results contained in the paper it is unfortunate that it is unnecessarily difficult to read because important steps are left out and a number of misprints appear. The following ones were especially troublesome to the reviewer. On p. 245, line 6 from bottom, replace u by t_{n_0-1}. On p. 248, line 10, replace y by $n_0y/(n_0+1)$. On p. 249, replace $x_i-c_i\xi$ wherever it appears (4 places near middle of page) by $x_i-d_i\xi$, where d_i is defined to be $c_i/(\sum_1^m c_j^2)^{\frac{1}{2}}$.]

H. Scheffé (Princeton, N. J.).

Mann, H. B. Note on a paper by C. W. Cotterman and L. H. Snyder. Ann. Math. Statistics 16, 311–312 (1945). [MF 13924]

It is shown that in the test proposed by Cotterman and Snyder to test Mendelian inheritance [J. Amer. Statist. Assoc. 34, 511–523 (1939)] the estimate used for the sample proportion of heterozygous individuals is for all practical purposes the one given by the method of maximum likelihood.

C. C. Craig (Ann Arbor, Mich.).

Finney, D. J. The fractional replication of factorial arrangements. Ann. Eugenics 12, 291–301 (1945). [MF 13927]

A certain set of treatment combinations is selected. The main effects and interactions are then estimated from these treatments. The effect of a factor a is estimated as the difference between the treatments containing a and those

not containing a. Similar expressions are formed for the interactions. A certain set A of effects and interactions can then not be estimated. If the treatment effects are considered as the elements of an Abelian group G, A forms a subgroup of G. If A contains r elements then every effect or interaction X is identified with all the elements of AX. Hence the effects are represented by the elements of the factor group G/A. The author also describes designs of this type in which some of the effects and interactions are confounded with blocks. *H. B. Mann* (Columbus, Ohio).

Daw, R. H. On the validity of statistical tests of the graduation of a mortality table. J. Inst. Actuar. **72**, 174–190, discussion 191–202 (1945). [MF 15103]

Vajda, S. The analysis of variance of mortality rates. J. Inst. Actuar. **72**, 240–245 (1945). [MF 15104]

Mathematical Economics

Kaplansky, Irving. A contribution to von Neumann's theory of games. Ann. of Math. (2) **46**, 474–479 (1945). [MF 13432]

The normalized form of a two-person zero-sum game is defined as follows: the first player A selects an integer i ($i=1, \cdots, m$) and the second player B selects an integer j ($j=1, \cdots, n$). After these choices have been made B pays the amount a_{ij} to A. A strategy of A is given by a vector $(p_1, \cdots, p_m)$ with $p_i \geqq 0$ and $\sum p_i = 1$. Similarly, a strategy of B is given by a vector $(q_1, \cdots, q_n)$ with $q_j \geqq 0$ and $\sum q_j = 1$. J. von Neumann has shown that there exist strategies $(p_1, \cdots, p_m)$ and $(q_1, \cdots, q_n)$ and a real value v' such that

$$(1) \qquad \sum_i p_i a_{ij} \geqq v', \qquad j=1, \cdots, n,$$

$$(2) \qquad \sum_j q_j a_{ij} \leqq v', \qquad i=1, \cdots, m.$$

The number v' is called the value of the game and strategies satisfying (1) and (2) are called good strategies. A strategy is pure if one component is equal to 1; otherwise it is mixed. If all components are positive, the strategy is said to be completely mixed. A game is said to be completely mixed if the only good strategies are completely mixed. The author shows that the value of a completely mixed game is given by $v' = |a_{ij}| / (\sum_{i,j} A_{ij})$, where A_{ij} is the cofactor of a_{ij}. If the game is not completely mixed, the author shows that at least one row or column can be deleted from $\|a_{ij}\|$ without changing the value of the game. This makes an inductive procedure possible for determining the value of an arbitrary two-person zero-sum game in a finite number of steps. *A. Wald* (New York, N. Y.).

Gheorghiu, Şerban. Sur la théorie de l'équilibre économique. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] **15**, 22–60 (1944). [MF 13580]

The author attempts to generalize the Walras-Pareto equilibrium solution by the addition of time as a determinant of the total utility function. Consider N coexchangists and let $\sigma_i, \alpha_i, \beta_i, \cdots, \lambda_i$ ($i=1, \cdots, N$) be the collection P_i of individual i at time t, with σ_i the means of payment and $\alpha_i, \beta_i, \cdots, \lambda_i$ the quantities of the n goods on the market. The total utility $U(P_i)$ is defined by $U(P_i) = \Phi(\sigma_i, \alpha_i, \beta_i, \cdots, \lambda_i; t)$; Φ is assumed capable of a Taylor expansion for any system of positive or zero values of the variables and t is assumed nonsingular. Let $p_1, \cdots, p_n$ be the money prices at time t of the n goods. Individual i will exchange P_i against the collection $P_i' = \{\sigma_i + s_i, \alpha_i + x_i, \cdots, \lambda_i + w_i\}$ under two conditions: (1) $U(P_i') = \text{maximum}$, (2) $s_i + p_1 x_i + \cdots + p_n w_i = 0$. The condition for (1) is easily found to be (3) $U_{\alpha_i}/1 = U_{\beta_i}/p_1 = \cdots = U_{\lambda_i}/p_n$, of which the n last terms express the familiar Walras hypothesis on marginal utility. Finally, since by exchange neither the quantity of money nor goods is altered, (4) $\sum_1^N s_i = \sum_1^N x_i = \cdots = \sum_1^N w_i = 0$. The systems (2), (3) and (4) provide the equilibrium solution, there being $(n+1)N+n$ equations and unknowns.

For the general equations of the dynamics of price it is initially assumed that the marginal utility of money is unity, that is, $U_{s_i} = 1$. Consider an interval Δt. Since consumption and/or production has taken place, at $t+\Delta t$ the collection of the ith individual is $\sigma_i, \alpha_i + \Delta\alpha_i, \cdots, \lambda_i + \Delta\lambda_i$. This necessitates a new exchange. Prices become $p_k + \Delta p_k$ and the equilibrium collection is $P_i^* = \{\sigma_i + \delta\sigma_i, \alpha_i + \Delta\alpha_i + \delta\alpha_i, \cdots, \lambda_i + \Delta\lambda_i + \delta\lambda_i\}$. Since the $\Delta\alpha_i, \cdots, \Delta\lambda_i$ are given, the unknowns of the problem are now the increments Δp_k and the $\delta\sigma_i, \delta\alpha_i, \cdots, \delta\lambda_i$. The necessary systems of equations for the unknowns are found. The equilibrium position at the time $t+\Delta t$ is determined by the total variations in the quantity of goods during Δt.

The (incomplete) stability condition is given by $\Delta p_1 = \cdots = \Delta p_n = 0$. Under the hypothesis that total utility is independent of time, this condition requires that $\sum_1^N \Delta\alpha_i = \cdots = \sum_1^N \Delta\lambda_i = 0$, a restriction which can be overcome. This stability condition is a necessary but not a sufficient condition for all markets, and a second definition of stability, similarly restricted, is offered to cover a special group of markets. In view of other recent discussions of stability conditions, the author's formulations are mathematically deficient. The author considers a number of other problems such as the solution of the "mean patrimony" and the time variability of the final utility of money. *M. P. Stoltz.*

Guillaume, Edouard. Sur la signification théorique des lois économiques statistiques mises en évidence par MM. Amstutz et Borloz. C. R. Séances Soc. Phys. Hist. Nat. Genève **60**, 79–83 (1943). [MF 14215]

This is an attempt to formalize certain empirical findings of Amstutz and Borloz [same C. R. **58**, 90–94 (1941); **60**, 57–74, 76–78 (1943)] relating to the production and stocks of gold and silver. Outside of the fact that the empirical results are conditioned by the method, the validity of the formal economic model is seriously compromised by the form of the equilibrium equation, wherein it is assumed that the values of the average product in different industries are equal. This can be true only under the extremely limited and unrealistic assumptions which make marginal and average quantities identical. *M. P. Stoltz.*

Hibbert, Lucien. Les équations du problème des fluctuations économiques et de l'interdépendance des marchés, d'après M. B. Chait. Bull. Soc. Math. France **69**, 1–22 (1941). [MF 13237]

The idea is expressed that the market is the fundamental unit of analysis of economic processes. The markets relevant to the production and distribution of any good are organized into sequences, and the sequences, in turn, into networks. In practice, any given sequence of markets is identified with a theoretical model. A market is specified by a "law of consumption" given by $Z_{i+1} = \lambda_i Z_i$, where

Z_{i+1} is the quantity of a good entering the market per unit time, Z_i is the quantity leaving the market per unit time and λ_i is a differential operator whose unknown parameters are determined statistically. Any sequence is specified by the successive application of the law of consumption to the markets of the sequence. The resulting differential equation takes the general form (*) $Z_p = F(Z_i)$, where Z_i is the outflow of the terminal market and Z_p the outflow of the initial market of the sequence. The ratio Z_p/Z_i, a general function of time, is termed the "law of divergence of markets." The determination of the unknowns Z_p and Z_i requires an additional equation expressing Z_i as a function of time, and is given by:

(**) $$g\beta+\sum h_s\beta^{(s)}-Z-uZ'=0,$$

where β is the margin of profit, g the quantity of production corresponding to unit profit, h_s the speculative factor, s the order of direct speculation, and u the acceleration of production per unit time; β is the difference between price and cost, which are represented as polynomial functions of the first degree in Z_k ($k=1, \cdots, i$). For a stable solution of (*) and (**) the roots of the characteristic equation should be complex. The method is extended to cyclical variations. There are numerous misprints.

M. P. Stoltz (Providence, R. I.).

Koopmans, Tjalling. Statistical estimation of simultaneous economic relations. J. Amer. Statist. Assoc. **40**, 448–466 (1945). [MF 15124]

TOPOLOGY

Bassi, Achille. On the importance of topology in modern mathematics. Gaz. Mat., Lisboa **6**, no. 26, 3–11 (1945). (Portuguese) [MF 15116]

Fréchet, Maurice. La notion d'uniformité et les écarts abstraits. C. R. Acad. Sci. Paris **221**, 337–340 (1945). [MF 14503]

A consideration of the merits of generalizing "metric" for topological spaces with retention of the condition that, of two "spheres" about the same point, one must contain the other. This condition is not retained in the theory of uniform structures due to A. Weil [Sur les espaces à structure uniforme et sur la topologie générale, Actual. Sci. Ind. no. 551, Hermann, Paris, 1938]. *R. Arens.*

Dieudonné, Jean. Sur les corps topologiques connexes. C. R. Acad. Sci. Paris **221**, 396–398 (1945). [MF 14681]

There exists a proper subfield S of the complex numbers, such that S forms a connected, everywhere dense subset of the plane. This strengthens a result of Livenson [Ann. of Math. (2) **38**, 920–922 (1937)]. The proof is a refinement of that of Livenson. *R. Arens* (Princeton, N. J.).

Kelly, Paul J. On isometries of square sets. Bull. Amer. Math. Soc. **51**, 960–963 (1945). [MF 14469]

It is an unsolved problem to establish or exhibit a counter-example for the conjecture, proposed by Ulam, that if two topological spaces R_1 and R_2 have the property that the Cartesian product $R_1\times R_1$ is homeomorphic to the Cartesian product $R_2\times R_2$, then the space R_1 is homeomorphic to the space R_2. The author establishes this conjecture for the case of certain isometries of a class of finite metric spaces. His proof employs a class of metrics (metrics of class α) for Cartesian products of finite metric spaces. Construction or existence of such metrics is not considered. The principal result is the following. Let E and F be finite metric spaces such that one of E or F has no self-isometry T, where T is not the identity and T^2 is the identity. Then if $E\times E$ and $F\times F$ are both metrized with metrics of class α and if $E\times E$ is isometric to $F\times F$, E is isometric to F. Two preliminary theorems on isometries of finite sets are also proved.

E. Hewitt (Princeton, N. J.).

***Ore, Oystein. Graphs and correspondences.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 184–191, Füssli, Zürich, 1945.

A mapping $a\to a^r$ of a set S into a subset is called a correspondence. A subset M of S is called closed whenever $a\in M$ implies $a^r\in M$ and vice versa. The set S decomposes into disjoint irreducible closed sets, called generalized cycles. A sequence $(a_0, \cdots, a_n)$ such that $a_{i+1}=a_i^r$, $a_0=a_n^r$ is called a finite cycle. Every generalized cycle contains at most one finite cycle. For each correspondence a directed graph is constructed. Necessary and sufficient conditions are given for a (directed or undirected) graph to be the graph of a correspondence. *S. Eilenberg* (Ann Arbor, Mich.).

Tietze, Heinrich. Über spezielle Simony-Knoten und Simony-Ketten mit vorgeschriebenen singulären Primzahlen. Monatsh. Math. Phys. **51**, 1–14 (1943). [MF 14377]

[For part II and notations see the same Monatsh. **51**, 85–100 (1944); these Rev. **6**, 280.] For an arbitrary finite set $\mathfrak{P}$ of odd primes and an arbitrary orientation there always exist an infinite number of torus knots and an infinite number of torus linkages $[2, n]$, which have exactly the numbers of $\mathfrak{P}$ as singular primes and which have the prescribed orientation. *H. Samelson* (Syracuse, N. Y.).

Hadwiger, H. Überdeckung einer Menge durch Mengen kleineren Durchmessers. Comment. Math. Helv. **18**, 73–75 (1945).

An elegant proof is given for the following conjecture of Borsuk [Fund. Math. **20**, 177–190 (1933)]: a set A in Euclidean n-space of diameter d can be covered by $n+1$ (but, in general, not by n) sets of diameters less than d.

H. Busemann (Northampton, Mass.).

Hadwiger, H. Ueberdeckung des Euklidischen Raumes durch kongruente Mengen. Portugaliae Math. **4**, 238–242 (1945). [MF 13363]

If Euclidean n-space is covered by means of $4n-3$ mutually congruent closed sets, each of them contains, for every positive δ, a pair of points at distance δ apart. The proof is effected by means of a covering lemma for an n-dimensional sphere. It is shown that for $n=2$, the theorem is false for 7 congruent closed sets; but it is not known whether it is true or false for 6 such sets. *H. Blumberg.*

Posniak, E. Sur les courbes fermées à tangentes parallèles. Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 59–64 (1945). (Russian. French summary) [MF 14593]

The author observes that, in sharp contrast to the state of affairs in the plane, a smooth closed curve situated in space need not possess even a single pair of parallel tangents.

He shows, among other things, that on an ellipsoid every smooth simple closed curve has at least one pair of parallel tangents. Along a slightly different line of inquiry, he develops some necessary conditions that a general space curve have at least one pair of parallel tangents. *L. Zippin.*

Bing, R. H. Generalizations of two theorems of Janiszewski. Bull. Amer. Math. Soc. **51**, 954–960 (1945). [MF 14468]

Suppose that neither H nor K separates the point A from the point B in the plane, $H \cdot K$ is connected or vacuous and H is compact. Then a theorem of Janiszewski asserts that $H+K$ will not separate A from B provided that H and K are compact continua. The author shows that any one of the following conditions can be substituted for the condition that H and K are compact continua: (1) H and K are closed in $H+K$, and $H \cdot K$ is a continuum or vacuous; (2) H and K are domains; (3) $H-H \cdot K$ and $K-H \cdot K$ are connected.

If H and K are compact continua and $H \cdot K$ is not connected, then $H+K$ cuts the plane [Janiszewski]. The author gives four generalizations of this theorem, one of which is as follows. The sum of two continuumwise connected sets separates the plane provided one of them is compact and their common part exists but does not belong to any continuumwise connected subset of their junction. (The junction of H and K is $H \cdot \bar{K}+\bar{H} \cdot K$.) *J. H. Roberts.*

Montgomery, Deane. Measure preserving homeomorphisms at fixed points. Bull. Amer. Math. Soc. **51**, 949–953 (1945). [MF 14467]

If T is a measure preserving automorphism of n-space E_n, A is a compact connected set such that $T(A) \subset A$, and U is an open set containing A, then there exists a compact connected set K containing A as a proper subset such that $T(K) \subset K$ and $K \subset T^{-1}(U)$. This is proved by reasoning similar to that used by Kerékjártó [C. R. Acad. Sci. Paris **198**, 1345–1347 (1934)]. If T has a fixed point, it follows that there exist continua, ranging from arbitrarily small to arbitrarily large, which include the fixed point and go into themselves under T. Analogous results hold for one-parameter continuous groups of measure preserving automorphisms of E_n. [The qualification "measure preserving" should be added to the hypothesis of theorem 2.]

J. C. Oxtoby (Bryn Mawr, Pa.).

Wallace, A. D. Generalized arc-sets. Proc. Nat. Acad. Sci. U. S. A. **31**, 414–417 (1945). [MF 14532]

This note announces a generalization along homotopy lines of the structure theory of continuous curves. Let H be a compact Hausdorff space. If X is a subspace (closed subset of H) all of whose mappings into the n-sphere S_n are inessential, X is a C_n; X is a T_n if each of its closed subsets is a C_n. A subspace is a B_n if it is not cut by any T_n and is maximal with respect to this property. If M is a subspace such that, for each subspace X and mapping $f : M \cdot X \to S_n$, there is an extension of f to X, then M is called an extension set in dimension n (or a J_n). Finally, a set N is an A_n if (1) N is a J_n, (2) if X is a B_n and $X \cdot N$ is not a T_n, then X is a subset of N.

The B_n and A_n sets are homotopy generalizations of cyclic elements and arc-sets of a continuous curve. It should be noted that these definitions are relative to the containing space H. Various structure theorems are stated. For example: the intersection of any number of A_n sets is an A_n; if $X+Y$ and $X \cdot Y$ are A_n sets, so are X and Y; if H satisfies a certain separation axiom, each A_n is an A_{n+1}; if H is a C_n but not a T_n, then it contains a nondegenerate B_{n-1}. The notion of an end-point is generalized as follows: a subspace E is a T_n-end-element if each neighborhood of E contains a neighborhood whose boundary is a T_n. Each T_n-end-element is an A_n. *E. G. Begle* (New Haven, Conn.).

Dedecker, Paul. Sur la notion d'involution et la formule de Zeuthen. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 680–687 (1943). [MF 13863]

Zeuthen's formula is generalized for an interior transformation of one closed surface onto another. The case of nonorientable surfaces is considered along with the case of orientable surfaces. Various extensions are indicated.

M. H. Heins (Cambridge, Mass.).

Chogoshvili, George. Théorème de dualité pour le polyèdre infini. C. R. Acad. Sci. Paris **221**, 15–17 (1945). [MF 14236]

The general duality theorem for a closed subset of the n-sphere and its complement, as given by Pontrjagin [Ann. of Math. (2) **35**, 904–914 (1934)], is extended to the case of a nonclosed set. The complement must still be an infinite polyhedron (homeomorphic image of a countable, locally finite Euclidean complex). The proof, which is outlined, involves the use of direct systems of groups whose limit groups are topologized. *E. G. Begle.*

Hirsch, Guy. Sur un théorème de Hopf-Rueff. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 516–524 (1943). [MF 13854]

Let M be a manifold which is a homology sphere (rational coefficients) and which is a fiber space over an orientable manifold Z with the fibers homeomorphic to an orientable manifold F. With each continuous mapping $T : M \to M$ carrying fibers into fibers are associated three integers: the degree c of T on M, the degree c' of the induced family of mappings $F \to F$, the degree c^* of the induced mapping $Z \to Z$. Two theorems are proved: first, if F is a homology sphere, then $c=0$ if and only if $c^*=0$; second, if Z is a homology sphere, then $c=0$ if and only if $c'=0$. The proofs are based on an unpublished theorem of the author concerning the algebraic number of fixed points. As an application, the author proves a theorem analogous to theorems of H. Hopf and M. Rueff [Comment. Math. Helv. **11**, 49–61 (1938)]. *S. Eilenberg* (Ann Arbor, Mich.).

Chern, Shiing-Shen. On Riemannian manifolds of four dimensions. Bull. Amer. Math. Soc. **51**, 964–971 (1945). [MF 14470]

Considering points in 4-space as quaternions, right and left translations by quaternions define orthogonal transformations. A frame, with its right translations, defines a "right Clifford figure"; similarly for left Clifford figures. Orienting them, the spaces R_r and R_l of right and left Clifford figures are homeomorphic to 3-spheres, and their products, to the orthogonal group G^4. At each point p of a Riemannian manifold M there are a tangent vector space, corresponding $G^4(p)$ and corresponding spaces $R_r(p)$ and $R_l(p)$; the latter form fibre bundles over M^4. The four-dimensional invariants I_r, I_l of these bundles are determined as integrals over M^4 of differential forms, and it is shown that $I_r-I_l=8\chi$ (χ is the Euler-Poincaré characteristic of M^4). Thus a new topological invariant $\eta=(I_r+I_l)/2$ is defined. Nothing is said about its nonvanishing or relation to other invariants. [Parts of the paper are obscure to

the reviewer; in particular, the statement and proof of theorem 2 (not used later, however). Take, for instance, $X=2^{-\frac{1}{2}}(1-i)$, $A=j$, $X'=k$; then $XX'=AX$; but if all quaternions are multiplied by $2^{-\frac{1}{2}}(1+i)$, the new determinant of $1, XX', AX, AX'$ is -1.] *H. Whitney.*

Hestenes, Magnus R. A theory of critical points. Amer. J. Math. 67, 521–562 (1945). [MF 13939]

In this paper an abstract theory of critical points is developed which is based on minimizing relative cycles (or "caps" in the terminology of Morse). [The theory is similar in its basis to the abstract theory of Morse.] There are given a metric space S and a class $\mathfrak{F}$ of subsets F of S having properties analogous to those possessed by the sets determined by $f(x)\leqq c$, where $f(x)$ is a lower semi-continuous function on S and c is a parameter. The relative cycles considered are Vietoris cycles u on F modulo F', denoted by (u, F, F'), where F, F' are in the class $\mathfrak{F}$ and $F'\subset F$. Minimizing cycles are introduced and the theory concerns itself with the study of these minimizing cycles, of their relation to the topology of the whole space S and their attachment to critical sets and critical points.

An equivalence is set up between minimizing k-cycles, the classes determined by this equivalence being called $\mathfrak{F}$-homology classes. Addition of $\mathfrak{F}$-homology classes is defined, and it is stated in theorem 9.1 that these form a group $\mathfrak{M}_k$. [This seems to be incorrect without further restriction, and so sections 9, 10, and 11 require serious modifications.] The $\mathfrak{F}$-homology classes determined by the same set F of $\mathfrak{F}$ form a group $\mathfrak{M}_k(F)$, and the following subgroups are defined: $\mathfrak{N}_k(F)$ consists of all $\mathfrak{F}$-homology classes in $\mathfrak{M}_k(F)$ containing an absolute k-cycle of the form $(u, F, 0)$; $\mathfrak{P}_k(F)$ consists of all $\mathfrak{F}$-homology classes in $\mathfrak{N}_k(F)$ containing an absolute k-cycle which is homologous to 0 on the whole space S. The following subgroups of the kth homology groups $\mathfrak{H}_k$ of the whole space S are introduced: $\mathfrak{H}_k(F)$ consists of those homology classes in $\mathfrak{H}_k$ containing a cycle on F; $\mathfrak{H}_k^*(F)$ consists of those homology classes in $\mathfrak{H}_k$ containing a cycle on a set F' in $\mathfrak{F}$, where F' is a proper subset of F. Various isomorphisms are set up, an example of which is

$$\mathfrak{N}_k(F)-\mathfrak{P}_k(F)\cong\mathfrak{H}_k(F)-\mathfrak{H}_k^*(F).$$

In case the coefficient group is a field and the dimensions of the relevant groups are finite, these isomorphisms yield the Morse relations. The paper closes with the application of the theory to functions of a finite number of variables and to single integral variational problems. *M. Shiffman.*

NUMERICAL AND GRAPHICAL METHODS

***Table of** arc sin x. Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards. Lyman J. Briggs, Director, National Bureau of Standards; Arnold N. Lowan, Director, Mathematical Tables Project. Columbia University Press, New York, 1945. xix+124 pp. $3.50.

The tables are to 12 decimal places; the tabular interval is .0001 in $0<x<.989$ and .00001 in $.989<x<1$. Second central differences are also given. Where the fourth difference becomes noticeable the now familiar "modified" differences are printed. This practical device enables one to obtain an accuracy of about 1.5 units using the simple Everett formula. Fourth order differences become necessary only for $.999<x<.9995$. For still larger x, ordinary interpolation becomes impossible; it is replaced by auxiliary tables of $\{\frac{1}{2}\pi-\arcsin(1-x)\}(2x)^{-\frac{1}{2}}$. Tables of interpolation coefficients, conversion tables and a table of multiples of $\frac{1}{2}\pi$ complete the book. *W. Feller* (Ithaca, N. Y.).

***Tables of Associated Legendre Functions.** Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards. Lyman J. Briggs, Director, National Bureau of Standards; Arnold N. Lowan, Director, Mathematical Tables Project. Columbia University Press, New York, 1945. xlvi+303 pp. $5.00.

The publication of this volume fills a long-felt gap in the existing literature. A particularly welcome feature is the inclusion of tables of Legendre functions of the second kind (for which there were no easily accessible tables other than the short table by Vandrey), Legendre functions with an imaginary variable (for which there seem to be no published tables), and of Legendre functions of half-integer order (tables for which were scanty and inadequate). The principal tables give $P_n^m(\cos\theta)$ and its first derivative for $n=1(1)10$, $m=0(1)4$, $\theta=0°(1°)90°$, $6S$; $P_n^m(x)$ and $Q_n^m(x)$ and their first derivatives for $n=1(1)10$, $m=0(1)4$, $x=1(.1)10$ and $-ix=0(.1)10$, $6S$, and $n=-.5(1)4.5$, $m=0(1)4$, $x=1(.1)10$, 4 to $6S$. (As usual, $a(b)c$ means that entries are given from a to b at intervals of c, and $6S$ stands for six significant figures, $6D$ for six decimals.) Supplementary tables give exact values of $P_n^m(x)$ for $m=0, 2, 4$; of $(x^2-1)^{\frac{1}{2}m}P_n^m(x)$ for $m=1, 3$; and 4 to $7D$ values of $(x^2-1)^{\frac{1}{2}}P_{n+1}^m(x)$ for $m=1, 3$, all for $x=1(.1)2$ (for interpolation purposes) with corresponding values for the derivatives; the normalizing factor for $n=0(1)10$, $m=0(1)10$, $n\geqq m$; and tables of coefficients of certain hypergeometric series. Further supplementary material includes an introductory account of the methods of computation, notes on interpolation in each of the principal tables, and a bibliography of 47 items, mostly tables (both published and manuscript). The volume is produced at the high standard for which the Mathematical Tables Project is so well known. *A. Erdélyi* (Edinburgh).

Seares, Frederick H. Trigonometric solution of the quadratic equation. Publ. Astr. Soc. Pacific 57, 307–309 (1945). [MF 14543]

The author is correct in surmising that the trigonometric solution given for the real roots of the quadratic equation $ax^2+bx+c=0$, based on the substitution $x=(c/a)^{\frac{1}{2}}\tan\theta$, is not new. [See, for example, Killing and Hovestadt, Handbuch des Mathematischen Unterrichts, vol. 2, Teubner, Leipzig, 1913, pp. 90–92.] The trigonometric solution of the quadratic is generally considered a curiosity. However, the author states: "The labor involved in the solution of these formulae is about two-thirds that required for a logarithmic calculation of the algebraic equation for the roots." *A. J. Kempner* (Boulder, Colo.).

Simonsen, W. On the numerical solution of systems of equations by means of iteration. Skand. Aktuarietidskr. 28, 154–170 (1945). [MF 14431]

The author intends to give a generalization of Newton's method, but what he really gives is a weaker, considerably more complicated method. It is weaker because the coeffi-

cients of the Newtonian system of linear equations are assumed to be constant; consequently the process, as is well known, converges only linearly instead of quadratically. It is more complicated because Newton's system consists merely of one system of n linear equations, while the author's system consists of n systems of n linear equations.
E. Bodewig (The Hague).

Lemaître, G. L'itération rationnelle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 347–354 (1942). [MF 13664]

Gauss applied a method of successive approximations to a system of two equations in two unknowns to determine the orbit of a planet from three observations. The paper contains a discussion of Gauss's ideas. Given $X(x, y)=0$, $Y(x, y)=0$ and tentative approximations x_n, y_n, $n=1, 2, 3$; let $X_n=X(x_n, y_n)$, $n=1, 2, 3$. It is assumed that these values are sufficiently close to the solution x', y' of the equations for them to be expressed linearly as $X_n=\alpha(x'-x_n)+\beta(y'-y_n)$, $Y_n=\gamma(x'-x_n)+\delta(y'-y_n)$. For this we must have

$$\begin{vmatrix} x'-x_1 & x'-x_2 & x'-x_3 \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{vmatrix}=0, \qquad \begin{vmatrix} y'-y_1 & y'-y_2 & y'-y_3 \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{vmatrix}=0$$

for the determination of x', y' as the next approximation. This is an "itération rationnelle," since it involves all earlier x_n, y_n. The author brings this problem into closer connection with ordinary iteration by assuming the equations in the form $f(x, y)=x$, $g(x, y)=y$. He applies these ideas to the solution of $f(x)=x$ and indicates their use for the Picard solution of $dy/dx=f(x, y)$ by iteration.

[Readers may be hesitant about accepting the evaluation [p. 351]: "Le principal avantage que présente l'itération rationnelle sur l'itération simple est qu'on ne doit pas s'inquiéter de mettre l'équation d'itération sous une forme telle que les valeurs x_1, x_2, x_3 obtenues successivement convergent vers la valeur reelle," since there is, as far as the reviewer sees, nothing to prevent the denominator in the next approximation $x_4=(x_1x_3-x_2x_3)/(x_1-2x_2+x_3)$ (for $f(x)=x$) from being very small as compared with the numerator.] *A. J. Kempner* (Boulder, Colo.).

Schürer, M. Die Keplersche Gleichung als Problem des wissenschaftlichen Rechnens. Astr. Nachr. **274**, 154–160 (1944). [MF 13995]

The first part of the paper contains an outline of the principles of scientific computation with references to F. Cohn and F. Klein. The second part illustrates these ideas by comparing several methods for solving the Kepler equation in the form $x=e\sin(M+x)$. The author grants that, given adequate computing machines, the standard iteration methods are as good as any. He then develops several series expansions for x, $\tan x$, $\sin x$, leading finally to what is referred to as "Encke's Verfahren":

$$x=\eta-\tfrac{1}{6}\cot M\eta^4+\tfrac{1}{6}\eta^5, \qquad \eta=\sin\left\{\tan^{-1}\frac{e\sin M}{1-e\cos M}\right\},$$

$$\tan x=\zeta-\tfrac{1}{6}\cot M\zeta^4+\tfrac{1}{6}\zeta^5, \quad \zeta=\sin\frac{e\sin M}{1-e\cos M},$$

$$\sin x=\xi-\tfrac{1}{6}\cot M\xi^4+\tfrac{3}{6}\xi^5, \quad \xi=\tfrac{1}{2}\sin\frac{2e\sin M}{1-e\cos M}.$$

Convergence questions are treated incidentally. Results are computed for $e=.3$, $M=50°$ and the accuracy of the various approximations is given. *A. J. Kempner.*

Blaskett, D. R., and Schwerdtfeger, H. A formula for the solution of an arbitrary analytic equation. Quart. Appl. Math. **3**, 266–268 (1945). [MF 13537]

This note contains simple proofs of the following results due to E. Schröder [Math. Ann. **2**, 317–363 (1870)]. Let α be a simple zero of an analytic function $f(z)$. Then, given a sufficiently good approximation z_0, we have

$$\alpha=z_0-\frac{f(z_0)}{f'(z_0)}-\frac{f(z_0)^2f''(z_0)}{2!f'(z_0)^3}+\frac{f(z_0)^3\{f'(z_0)f'''(z_0)-3f''(z_0)^2\}}{3!f'(z_0)^5}+\cdots.$$

Here the general term is of the form

$$u_\nu(z)=(-1)^\nu f(z_0)^\nu f_\nu(z_0)/\nu!,$$

where $f_0(z)=z$, $f'(z)f_{\nu+1}(z)=f_\nu'(z)$. If we define $\Phi_n(z)$ by $\Phi_n(z)=\sum_{\nu=0}^{n}u_\nu(z)$, then replacing z_0 by $\Phi_1(z_0)$ is the first step in Newton's method. In general, one may construct a sequence α_m defined by $\alpha_0=z_0$, $\alpha_{m+1}=\Phi_n(\alpha_m)$; then (1) $\lim_{m\to\infty}\alpha_m=\alpha$, (2) $f(\alpha)=0$ so that $\alpha=\Phi_n(\alpha)$ and (3) $\Phi_n'(\alpha)=\Phi_n''(\alpha)=\cdots=\Phi_n^{(n)}(\alpha)=0$. *D. H. Lehmer.*

Waugh, Frederick V., and Dwyer, Paul S. Compact computation of the inverse of a matrix. Ann. Math. Statistics **16**, 259–271 (1945). [MF 13916]

Continuing and greatly improving their previous work, the authors describe simplified ways of computing, in compact form, the inverse of an unsymmetric matrix. Use is made, in particular, of a factorization of the familiar auxiliary matrix. It is not possible to describe the method adequately in a brief review. *W. Feller* (Ithaca, N. Y.).

Vernotte, Pierre. Sommation, par des représentations empiriques, des séries lentement convergentes rencontrées en physique mathématique (rectification de valeurs classiques incorrectes). C. R. Acad. Sci. Paris **218**, 67–69 (1944). [MF 13475]

The author describes a method of approximate summation of $\sum u_n$, based on representing u_n as a sum of terms n^{-s}. In this connection he gives $\zeta(5)$ and $\zeta(7)$ to 24 and 25 decimals, respectively, "correcting the classical values." While the latter are not specified, they are presumably those of Euler or Legendre. However, the last few places given by the author are in disagreement with the tables of Stieltjes and others [references in H. T. Davis, Tables of the Higher Mathematical Functions, vol. 2, Principia Press, Bloomington, Ind., 1935, p. 217]. *I. Kaplansky* (Chicago, Ill.).

van Veen, S. C. Convergence possibilities in the application of elliptic integrals. Handelingen van het XXIXe Nederlandsch Natuur- en Geneeskundig Congres, Amsterdam, 1943. 6 pp. (Dutch) [MF 14298]

Bordoni, Piero Giorgio. The conical sound source. J. Acoust. Soc. Amer. **17**, 123–126 (1945). [MF 13794]

W. N. Brown [same J. **13**, 20–22 (1941)] calculated the acoustic pressure due to a rigid cone axially vibrating in an infinite baffle. In his expression there appears the integral

$$\int_0^\infty e^{ias}J_0(gs)sds.$$

The author expands this integral in the infinite series

$$a^2\sum_{n=0}^{\infty}(iab)^nG_n(ag)/n!,$$

where

$$G_n(z)=z^{-n-2}\int_0^z x^{n+1}J_0(x)dx.$$

He gives explicit expressions for $G_n(z)$ for $n=0, 1, \cdots, 6$, numerical tables for $n=0(1)6$ and $z=0(1)5$ to three (partly four) decimals, and states that for $n>6$ and $0<z<5$ $G_n(z)=(n+2)^{-1}J_0(z(n+2)/(n+3))$, approximately. The series obtained has a physical meaning in that the first term represents the acoustic pressure due to a flat disc of the same radius as the base of the cone and the later terms give corrections due to the conical shape of the sound source. Graphs are given of the pressure on the axis and also of some directional curves. The results "give a mathematical proof that, in most technical work, a conical radiator can be approximated by a flat disc of equal radius."

A. Erdélyi (Edinburgh).

Salzer, Herbert E. Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs. J. Math. Phys. Mass. Inst. Tech. **24**, 141–143 (1945). [MF 15129]

Coefficients are given for interpolation in Lagrange form of a function of a complex variable by means of a polynomial of degree $n-1$ ($n=3, 4, 5$) passing through the values of the function at n points equally spaced on the arc of a circle in the complex plane. Separate expressions are given for the real and imaginary parts of the coefficients in terms of the real and imaginary parts of the given values of the function and the angle between successive chords joining the given points. The adaptation of the formulas for inverse interpolation is also explained. *T. N. E. Greville.*

Salzer, Herbert E. Coefficients for mid-interval numerical integration with central differences. Philos. Mag. (7) **36**, 216–218 (1945). [MF 13346]

The author gives the coefficients K_{2s} in the first Gaussian summation formula for numerical integration. With the integrand $f(x)$ tabulated at equidistant intervals h, this formula yields

$$h^{-1}\int_{a-\frac{1}{2}h}^{a+(n-\frac{1}{2})h} f(x)dx,$$

where a is a tabular argument, so that upper and lower limits of the integration fall at midpoints of tabular intervals. The expansion is in terms of central differences of even order ($2s$) and the coefficients K_{2s} [hitherto known for $2s=0(2)8$] are given as exact fractions for $2s=0(2)20$ and to 18 places of decimals for $2s=22(2)50$. [In practice the coefficients beyond $2s=6$ are rarely used.]

H. O. Hartley (London).

Steffensen, J. F. On certain formulas of mechanical quadrature. Skand. Aktuarietidskr. **28**, 1–19 (1945). [MF 14152]

The chief purpose of the paper is to modify the integration formulae of Gauss and Chebyshev by replacing the irrational arguments by neighboring rational ones for the sake of convenience without an appreciable loss of accuracy. This is attained by integrating Lagrange's interpolation formula with remainder and specializing the available constants so that the remainder of the resulting integration formula will be as small as possible when the number n of arguments is prescribed. The procedure is used in practice only for $n=2, 3, 4, 5$. So, for example, with $f(\pm x)\equiv f(x)+f(-x)$ and $|\xi|<1$, the integral J taken from -1 to $+1$ will be given by the formulae, almost as exact as Chebyshev's,

$$1.47J=.94f(0)+f(\pm.7)+.0044f^{iv}(\xi);$$
$$45J=23f(\pm.2)+22f(\pm.8)-.004f^{iv}(\vartheta)+.002f^{vi}(\xi),$$

where $|\vartheta|<.8$; $J=Af(0)+Bf(\pm 3/8)+Cf(\pm 5/6)+Kf^{vi}(\xi)$, where $A=1334/3375$, $B=17408/43065$, $C=15876/39875$, $K=.0^424$; or by the formula, practically of the same approximation as Gauss's,

$$49J=44f(0)+27f(\pm 7/9)-.0069f^{iv}(\vartheta)+.003f^{vi}(\xi),$$

where $|\vartheta|<7/9$; or by the formulae of the author,

$$6J=5f(\pm\sqrt{\tfrac{1}{5}})+f(\pm 1)-.0005f^{vi}(\xi);$$
$$15J=8f(0)+9f(\pm\sqrt{\tfrac{1}{5}})+2f(\pm 1)-.00054f^{vi}(\xi);$$

or, with nearly the same approximation,

$$65J=54f(\pm 4/9)+11f(\pm 1)-.0091f^{iv}(\xi_1)-.0056f^{vi}(\xi_2),$$
$$3960J=2046f(0)+2401f(\pm 4/7)+536f(\pm 1)-.15f^{vi}(\xi).$$

E. Bodewig (The Hague).

Bottema, O. Simpson's method of approximation. Nieuw Arch. Wiskde. (2) **21**, 111–118 (1941). (Dutch) [MF 14297]

Let $T(n)$ denote the "trapezoidal" approximation to $I=\int_a^b f(x)dx$, corresponding to a division of (a, b) into n equal parts. By applying the Euler-Maclaurin summation formula to each part and combining the results, the author shows that, if $f(x)$ has a continuous $(2k+1)$th derivative in the closed interval ($k\geqq 1$), and if $f^{(j)}(a)=f^{(j)}(b)$ for $j=1, 3, \cdots, 2k-3$, then

$$\lim_{n\to\infty} n^{2k}(T(n)-I)=B_{2k}(b-a)^{2k}\{f^{(2k-1)}(b)-f^{(2k-1)}(a)\}/(2k)!,$$

where B_{2k} is a Bernoulli number. By applying this result, he obtains similar relations for $S(n)=(4/3)T(2n)-(1/3)T(n)$, the Simpson approximations with $2n$ subintervals, and for $N(n)=(9/8)T(3n)-(1/8)T(n)$, the Newton approximations; in these cases, $k\geqq 2$, but $f^{(j)}(a)=f^{(j)}(b)$ only for $j=3, 5, \cdots, 2k-3$. If no restriction at all is placed on $f^{(j)}(a)$, $f^{(j)}(b)$, we have $T(n)-I=O(n^{-2})$, $S(n)-I=O(n^{-4})$, $T(n)-I=O(n^{-4})$, where the error terms involve $f'(b)-f'(a)$ for $T(n)$, $f'''(b)-f'''(a)$ for $S(n)$ and $N(n)$.

R. P. Boas, Jr. (Providence, R. I.).

Kneschke, A. Über die genäherte Quadratur. Monatsh. Math. Phys. **51**, 15–23 (1943). [MF 14378]

The author discusses a class of quadrature formulas of the type

$$\int_a^b f(x)dx=\sum_{\mu=0}^{n}\{A_{\mu 0}f(x_\mu)+A_{\mu 1}f'(x_\mu)+\cdots+A_{\mu m}f^{(m)}(x_\mu)\}+R,$$

$$R=\sum_{\nu=1}^{n}\int_{x_{\nu-1}}^{x_\nu} f^{(m+1)}(\xi)N_\nu(\xi)d\xi,$$

where the N_ν are given polynomials and

$$A_{\mu k}=(-1)^{m-k}\{N_{\mu+1}^{(m-k)}(x_\mu)-N_\mu^{(m-k)}(x_\mu)\}.$$

For special choices of the N_ν, generalizations of classical formulas are obtained. *R. P. Boas, Jr.*

Vernotte, Pierre. Détermination, par la condition de moindre imprécision, du polynome du second degré représentant au mieux l'ensemble d'une courbe expérimentale. C. R. Acad. Sci. Paris **221**, 609–611 (1945). [MF 15149]

Nyström, E. J. Zur numerischen Lösung von Randwertaufgaben bei gewöhnlichen Differentialgleichungen. Acta Math. 76, 157–184 (1945). [MF 13199]

The author develops a numerical method of solving $y''=f(x, y)$ for $-\frac{1}{2}\leqq x\leqq+\frac{1}{2}$ with boundary conditions $y(-\frac{1}{2})=y(\frac{1}{2})=0$. The general case of the first boundary value problem is reduced to this special case by a linear transformation of y and x. The starting point is the equivalent integral equation

$$(1)\qquad y(x)=\int_{-\frac{1}{2}}^{+\frac{1}{2}}K(x, t)f(t, y(t))dt,$$

where the kernel $K(x, t)$ is Green's function for the problem. The method consists in a representation of $\int_{-\frac{1}{2}}^{+\frac{1}{2}}K(x_j, t)f(t)$ in the form $\sum_{i=0}^{K}{}_jG_i f(t)$, where the weights ${}_jG_i$ are determined in such a way that for $K+1$ points $x=x_j$ the representation is exact for any polynomial $f(t)$ of order K. Equidistant patterns of 1, 2, 3 and 4 internal points x_j ($K=2, 3, 4$ and 5) are considered, each of which gives rise to a different matrix of weights ${}_jG_i$. These weights are tabulated in an appendix and are applicable to solving (1) for any $f(x, y)$ for which $f(x, y(x))$ can be approximated by a Kth order polynomial to sufficient accuracy.

The numerical work consists of solving the equations $y_i=\sum {}_jG_i f(x_i, y_i)$ for the $K+1$ unknowns y_j; these are linear if f is a linear function of y. Iteration is recommended for their solution in the general case. From the $K+1$ values of y_j the solution $y(x)$ is to be obtained for $-\frac{1}{2}\leqq x\leqq+\frac{1}{2}$ by polynomial (Lagrangian) interpolation. [This clearly limits the method to equations with "smooth" solutions $y(x)$ which can be obtained by interpolating between 6 or fewer pivotal points x_j, y_j.] The order of approximation is improved by producing weights ${}_jG_i$ for specially selected patterns of non-equidistant x_j.

The method is illustrated by numerous examples, one of which covers the generalisation to a system of two second order equations. The second and third boundary value problems can be treated in a similar manner.

H. O. Hartley (London).

Mikheladze, Š. A new method for the calculation of the trajectory of the center of gravity of a projectile. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 861–865 (1944). (Georgian. Russian summary) [MF 14616]

Mikheladze, Š. Application of a formula of numerical differentiation to the calculation of the trajectory of the center of gravity of an artillery shell. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 959–964 (1944). (Georgian. Russian summary) [MF 14617]

For the method of numerical integration used, the author refers to his papers in Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 627–642 (1939); C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 400–402 (1940); these Rev. **2**, 63, 198.

Mikheladze, Š. On a method of numerical differentiation. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 663–666 (1944). (Georgian. Russian summary) [MF 14609]

A formula for numerical differentiation given by Steffensen ["Interpolation," Williams and Wilkins, Baltimore, 1927, p. 64, formula (16)] is valid only when the variable x is outside the interval bounded by the smallest and largest of the points a_k involved in the differences used. This restriction is pointed out in the book, but apparently not in the Russian edition of 1935 to which the author refers. He establishes the formula under this restriction and shows that the restriction is essential. *R. P. Boas, Jr.*

Rossier, Paul. Construction du cercle osculateur à une courbe graphique au moyen d'une courbe d'erreur. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 25 (1945). [MF 14226]

Artmeladze, N. K. On the approximate solution of integral equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 13, 29–53 (1944). (Russian. Georgian summary) [MF 14619]

The object of this paper is to develop new methods for the approximate numerical solution of integral equations of the Fredholm type, in one and two dimensions, both with regular kernels and with kernels with singularities. The method consists in subdividing the region of integration into rectangles and right triangles. The integral is then replaced by approximating finite sums over these subregions, and the integral equation is replaced by a system of linear equations in which the unknowns are the values of the unknown function at a discrete set of points. Sufficient conditions for the validity of this method are given. A number of examples are worked out in detail for equations for which the exact solution is known, and the percent of error between the exact and approximate solutions is computed. These errors vary in the given examples from two to nine percent. *H. P. Thielman* (Ames, Iowa).

Hildebrand, Francis B. A least-squares procedure for the solution of the lifting-line integral equation. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **925**, 37 pp. (4 plates) (1944). [MF 14398]

The usual procedure for obtaining a numerical solution of the Prandtl lifting-line equation consists in assuming that the solution can be approximated by a finite series of appropriate functions and requiring that the equation is satisfied at N points. This leads to a system of N linear algebraic equations with the N coefficients as unknowns. The author suggests an alternative procedure in which the unknown coefficients are determined from the condition that the integral equation is approximated at M points ($M\geqq N$) so that the sum of the squares of the errors multiplied by appropriate weighting coefficients is a minimum. This method is also applied to cases where the angle of attack function and the chord function, which occur as coefficients in the lifting-line equation, are discontinuous. The computation is mechanical and is said to require considerably less time than other methods of comparable accuracy. *M. Golomb* (Philadelphia, Pa.).

Germain, Paul. Sur le calcul numérique de certains opérateurs linéaires. C. R. Acad. Sci. Paris **220**, 765–768 (1945). [MF 14067]

The note deals with a numerical method for evaluating operators A which transform functions $P(\theta)$ of period 2π into functions $Q(\theta)$ of the same period and which have the property that either

$$A(\cos m\theta)=b_m\sin m\theta,\quad A(\sin m\theta)=-a_m\cos m\theta,\quad A(1)=0$$

or

$$A(\cos m\theta)=b_m\cos m\theta,\quad A(\sin m\theta)=b_m\sin m\theta,\quad A(1)=b_0$$

where the a_m and b_m are given constants. If $P(\theta)$ is a trigonometric polynomial of degree n, $Q(\theta)$ is computed by finding the values $Q(k\pi/n)$ as linear functions of the values $P(k\pi/n)$ or by finding the values $Q(2k\pi/n)$ as linear functions of the values $P((2k-1)\pi/n)$, for $k=1, \cdots, 2n$. For arbitrary functions $P(\theta)$ the same formulas represent suitable approximations. *M. Golomb* (Philadelphia, Pa.).

Flamache, L. Conditions d'anamorphose de certaines relations trinômes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 314–320 (1943). [MF 13848]

A function of three variables z_1, z_2, z_3 may be represented by an alignment chart if and only if it has the form $|f_i \;\; g_i \;\; h_i| = 0$. The elements f_i, g_i, h_i of this determinant denote functions of only one variable z_i $(i=1, 2, 3)$. A relation $Ff_3+Gg_3+Hh_3=0$, where F, G, H depend only on z_1 and z_2, is considered. The author shows that an alignment chart for this relation exists if and only if

$$\begin{vmatrix} F & G & H \\ F_{z_1} & G_{z_1} & H_{z_1} \\ F_{z_2} & G_{z_2} & H_{z_2} \end{vmatrix} \neq 0, \quad \begin{vmatrix} F & G & H \\ F_{z_i} & G_{z_i} & H_{z_i} \\ F_{z_iz_i} & G_{z_iz_i} & H_{z_iz_i} \end{vmatrix} = 0$$

for $i=1, 2$. *E. Lukacs* (Cincinnati, Ohio).

Nikolaev, P. V. On the anamorphosis of symmetric equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 82–86 (1945). [MF 13744]

Nikolaev, P. V. The rational anamorphosis of equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 155–158 (1945). [MF 13734]

These papers are a continuation of the author's previous work on anamorphosibility of polynomials [cf. the same C. R. (N.S.) **28**, 582–584, 774–777 (1940); these Rev. **2**, 240]; they both concern the anamorphosibility of an irreducible algebraic equation $F(t_1, t_2, t_3)=0$, that is, the existence of a polynomial

$$\Phi(t_1, t_2, t_3) \equiv \psi_1(t_2, t_3)\psi_2(t_3, t_1)\psi_3(t_1, t_2)F(t_1, t_2, t_3)$$

which can be identically represented in the form of a Masseau determinant $|f_{ij}(t_i)|$ $(i, j=1, 2, 3)$, where each f_{ij} is a polynomial in t_i.

In the first paper the author considers the general theory of the case in which the equation is symmetrical with respect to t_2, t_3; he then obtains, in particular, sets of necessary and sufficient conditions for an equation to admit of one or other of the anamorphosizing factors (i) t_2-t_3, (ii) $(t_2-t_3)(t_3-t_1)(t_1-t_2)$.

In the second paper, the author considers the general case and sets out a detailed step-by-step procedure for investigating the anamorphosibility or non-anamorphosibility of any given equation. *J. G. Semple* (London).

Nicolaeff, P. Les transformations rationnelles d'un nomogramme. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 83–98 (1944). (Russian. French summary) [MF 15196]

Nicolaeff, P. Polynomes de Masseau. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 99–116 (1944). (Russian. French summary) [MF 15197]

Nikolaev, P. V. On the uniqueness of anamorphoses of Masseau equations. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 117–128 (1944). (Russian) [MF 15198]

Cf. C. R. (Doklady) Acad. Sci. URSS **28**, 582–584, 774–777 (1940); these Rev. **2**, 240; and the preceding review.

Malavard, Lucien. Calculateur d'ailes et réseaux de résistances linéaires pouvant remplacer, dans certaines questions, le bassin électrique. C. R. Acad. Sci. Paris **221**, 224–226 (1945). [MF 14253]

The apparatus in question can be used to solve systems of linear algebraic equations.

Gutenmacher, L. I. Electric circuits for approximate solution of system of equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 259–262 (1945). [MF 14027]

Gutenmacher, L. I. Artificial many-dimensional models for approximate solutions of integral equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 94–96 (1945). [MF 13747]

Electric networks similar to those that the author used to model partial difference equations [same C. R. (N.S.) **27**, 198–202 (1940); these Rev. **2**, 240] are proposed as models for integral and integro-differential equations, the integrals being approximated by finite sums. A number of two-terminals consisting of arbitrary combinations of lumped resistances, inductances and capacitances are connected at junction points of an n-dimensional lattice and each of the junction points is connected to ground by a lumped conductance, which may contain a source of e.m.f. The voltages at the junction points satisfy a system of linear equations which converge to a system of n linear integro-differential equations as the number of junction points increases indefinitely. *M. Golomb.*

Gutenmacher, L. I. The integral equations of many-dimensional electric models. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 169–171 (1945). [MF 13738]

Specifying the e.m.f. sources mentioned in the paper reviewed above to be alternating of frequency ω and the transfer impedances of the lattice network and the conductances to ground to be purely reactive, the steady-state voltages at the junction points become approximate solutions of a system of Fredholm equations with ω as parameter. For certain values of ω the circuits fall into resonance. These values correspond to the eigenvalues of the Fredholm system. If there is energy stored in the system and there is no external e.m.f. source, characteristic vibrations take place in the circuits; these correspond to the characteristic solutions of the Fredholm system. If the transfer impedances and conductances to ground are purely resistive, then the kernel of the modeled integral equation becomes positive definite and independent of ω. There can be no resonance, and the homogeneous equation has none but the trivial solution. If the circuits contain both reactive and resistive elements the modeled equations become more involved. *M. Golomb* (Philadelphia, Pa.).

Goussinsky, B. Zum Problem der algebraischen Multiplikation mittels der Rechenmaschine. Schweiz. Z. Vermessgswes. Kulturtech. **42**, 251–252 (1944). [MF 13997]

Hansen, W. W., and Lundstrom, O. C. Experimental determination of impedance functions by the use of an electrolytic tank. Proc. I.R.E. **33**, 528–534 (1945).

The impedance of a lumped element electric network is a rational function of the complex radian frequency ω and its logarithm therefore has the form

$$\log |Z| = \log |a| + \sum \log |\omega-\omega_s| - \sum \log |\omega-\omega_p|,$$

ω_s and ω_p being the zeros and poles of Z. This is analogous

to the formula for potential in plane potential theory. Hence, by placing electrodes of equal positive and negative voltage at the poles and zeros in an electrolytic tank, $\log|Z|$ can be measured as the voltage at any point ω. In this paper the authors develop the details of this method and give some experimental results. It is shown that the error in using a finite tank is small and can be corrected. Methods of determining the phase of Z are given.

C. E. Shannon (New York, N. Y.).

Hain, Kurt. Die Verwendung der sechsgliedrigen Gelenkgetriebe als Rechengetriebe. Z. Instrumentenkunde **64**, 96–104 (1944). [MF 13628]

Certain types of six-bar linkages are studied, in particular, linkages obtained by combining two four-bar linkages so that the driven crank of the first is rigidly connected with the driving shaft of the second. The author discusses the problem of constructing linkages to mechanize a prescribed motion. The paper continues the author's study of four-bar linkages [same Z. **63**, 170–180 (1943); these Rev. **5**, 246]. *W. Feller* (Ithaca, N. Y.).

Hunziker, Ad. Compensation d'un réseau de nivellement. Schweiz. Z. Vermessgswes. Kulturtech. **42**, 241–247 (1944). [MF 13996]

Vignal, Jean. Compensation d'un réseau de nivellements. C. R. Acad. Sci. Paris **220**, 352–354 (1945). [MF 13948]

Appliquée à la compensation d'un réseau de nivellements, la règle classique des moindres carrés conduit, dans quelques-uns de ses modes d'application, à des systèmes d'équations de forme particulièrement simple, susceptibles d'être écrits sans aucun calcul préparatoire. *Extract from the paper.*

Ruffet, Jean. L'aplatissement terrestre calculé en seconde approximation. Arch. Sci. Phys. Nat., Geneva **25**, 83–90 (1943). [MF 14212]

Ferrieu, François. Étude sur les mesures liées des grandeurs: Problème des moyennes. Rev. Sci. (Rev. Rose Illus.) **81**, 203–216 (1943). [MF 13988]

Andersen, N. E. The valuation of the temporary annuity $\bar{a}_{\overline{n}|}-\bar{a}_{\overline{x:n}|}$. Skand. Aktuarietidskr. **28**, 218–227 (1945). [MF 14435]

The author considers a temporary annuity, with payments due from the death of the person insured until the expiration of a set period. It is shown that the premium reserves for such an annuity, on the basis of a constant rate of interest and a nondecreasing force of mortality, are positive during the entire insurance period provided that the premium payment period is half the insurance period and ${}_np_x>(1+i)^{-n}$. Here x is the age at entry, n the premium payment period, i the rate of interest. *E. Lukacs.*

Johansen, Paul. Approximate valuation of joint life policies. Skand. Aktuarietidskr. **28**, 171–180 (1945). [MF 14432]

Trier, Gunnar. On some methods of single-premium apportionment, especially in collective pensions insurance. Skand. Aktuarietidskr. **26**, 1–14 (1943). [MF 14140]

* **Jecklin, Heinrich. Über eine Näherungsformel der Versicherungstechnik.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 111–117, Füssli, Zürich, 1945.

Let $\Delta_{x\overline{n}|}=P_{x\overline{n}|}-P_{\overline{n}|}=1/a_{x\overline{n}|}-1/a_{\overline{n}|}$ be the "mortality component" of the net premium for an endowment insurance. The Hölder-Jensen inequality gives an upper bound for $\Delta_{x\overline{n}|}$, whereas $1/e_{x\overline{n}|}-1/n$ is a lower bound, valid for the ages offering practical interest. The author uses the arithmetic mean of these bounds as an approximation; thus he obtains

$$\Delta_{x\overline{n}|}\sim Z_{x\overline{n}|}=\left(l_x/\sum_0^{n-1}l_{x+t}-1/n\right)(1+\tfrac{1}{2}in).$$

Here i denotes the rate of interest, n the duration, l_x the number living at age x. Using this formula it is possible to approximate various actuarial functions without using commutation columns; the life table and interest table suffice for carrying out the computations. *E. Lukacs.*

Steffensen, J. F. On the construction of tables for the calculation of certain survivorship benefits. Skand. Aktuarietidskr. **27**, 154–171 (1944). [MF 14148]

The author considers a special form of life insurance which he calls a death annuity $\bar{a}_{x||y}$. If x is the age of the person insured and y the age of the beneficiary,

$$\bar{a}_{x||y}=\int_0^\infty v^t{}_tp_x\mu_{x+t}a_{y+t}dt.$$

This is a decreasing life insurance where the sum insured t years after entry is a_{y+t}. It is assumed that the annuities on the life of the beneficiary may be graduated according to the formula $a_y=Aa^y+Bb^y+Cc^y$. The six constants A, B, C, a, b, c are determined by requiring that the graduated values shall agree with the observed values for 6 equidistant ages. If this graduation is used, the death annuity assumes the form

$$\bar{a}_{x||x-h}=f_1(x)g_1(h)+f_2(x)g_2(h)+f_3(x)g_3(h),$$

where $h=x-y$ is the age difference. This form avoids the use of tables with two entries and facilitates the work of the computer. The author uses Danish life tables to show that the graduation is good enough for practical purposes and computes tables for the auxiliary functions.

E. Lukacs (Cincinnati, Ohio).

Hagstroem, K.-G. The general life assurance. Skand. Aktuarietidskr. **24**, 137–158 (1941). [MF 14131]

A general life insurance on several lives is discussed by an extension of A. Loewy's methods [S.-B. Heidelberger Akad. Wiss. Abt. A. **1917**, no. 6]. Each possible composition of the group of insured lives is called a phase; only contracts with a finite number of phases are considered. Reserve functions satisfying an equation of equilibrium are introduced and determined recurrently. Finally, a "modulus" is defined; it is proposed to use it as a measure of the insurance protection offered by the contract. *E. Lukacs.*

Simonsen, W. On the construction of biometric functions from the experience of life insurance companies. Skand. Aktuarietidskr. **26**, 145–175 (1943). [MF 14142]

Starting from the fundamental notion of intensities, relations involving intensities as well as observable quantities are derived. Suitable approximations are then applied to deduce formulae generally used in constructing biometric functions. This is done for aggregate and select mortality tables and extended to invalidity functions. The possibility of reactivity is taken into account and select as well as aggregate intensities for the disabled are discussed.

E. Lukacs (Cincinnati, Ohio).

MECHANICS

Leif

***Brelot, M. Les Principes Mathématiques de la Mécanique Classique.** B. Arthaud, Grenoble-Paris, 1945. 62 pp.

An attempt to give a rigorous summary of rational mechanics, making use of additive sets, existence theorems for differential equations, and similar results of analysis.

P. Franklin (Cambridge, Mass.).

Simons, Lennart. Geschossabweichungen infolge der Erddrehung unter Berücksichtigung des Luftwiderstandes. Soc. Sci. Fenn. Comment. Phys.-Math. 12, no. 11, 12 pp. (1944). [MF 14628]

The author sets up the differential equations for variations of trajectories due to rotation of the earth, including the terms involving the air resistance function. He works out tables for some specific cases, based on Garnier's resistance function and assuming a constant ballistic coefficient.

P. Franklin (Cambridge, Mass.).

Bottema, O. The motion of a material point in a rotating tube. Nieuw Arch. Wiskde. (2) 21, 12–24 (1941). (Dutch) [MF 14294]

van Veen, S. C. Elementarer Beweis eines Halphenschen Satzes über die Bewegung des sphärischen Pendels. Nieuw Arch. Wiskde. (2) 21, 119–123 (1941). [MF 14301]

Proof of Halphen's result that the azimuth distance between two successive vertices of the orbit of a spherical pendulum is between $\pi/2$ and π, without use of elliptic integrals.

P. Franklin (Cambridge, Mass.).

Giraud, Georges. Petits mouvements à courte période et petits mouvements amortis d'une masse tournante composée d'un liquide homogène et d'un noyau solide immergé. Ann. Sci. École Norm. Sup. (3) 58, 37–82 (1941). [MF 12898]

Consider a rotating body composed of a solid kernel completely immersed in a homogeneous liquid. Suppose that this body is in a position of relative equilibrium under the combined influence of centrifugal force and the mutual gravitational attraction between the particles. Besides these forces the body is influenced by a small disturbing force with a potential $V(x, y, z)e^{\lambda t}$, where λ is an arbitrary complex number. The author studies the possibility of either small periodic or small damped motions in the neighborhood of the equilibrium position. The problem is reduced to the solution of a set of equations linear in seven unknowns, six of which are constants while the seventh is a function ψ defined in the region occupied by the liquid in its equilibrium position and satisfying the equation of Poincaré, $\psi_{zz}+\psi_{yy}+(1-4\omega^2/\lambda^2)\psi_{xx}=0$, where ω is the angular velocity of the body. A method is given for solving the equations in the case where λ belongs to the set of points in the complex plane whose distance from the interval $(-2\omega, 2\omega)$ is greater than zero.

B. Friedman (New York, N. Y.).

Stueckelberg, Ernest-C.-G. Principe de correspondance d'une mécanique asymptotique classique. C. R. Séances Soc. Phys. Hist. Nat. Genève 61, 155–158 (1944). [MF 14223]

Johnsen, ~~Lief.~~ Dynamique générale des systèmes nonholonomes. Skr. Norske Vid. Akad. Oslo. I. 1941, no. 4, 75 pp. (1941). [MF 14160]

This paper gives an almost complete account of the general theory of nonholonomic dynamical systems, starting out in the introductory chapter with the derivations of the various forms of the known equations of motion as written by Lagrange, Hamel, Woronetz, and Appell. The author's most distinctive contribution (in chapters III and IV) is his generalization of the symbolic calculus of pseudo-coordinates so as to take care of the case when the velocities enter nonlinearly into the equations of constraint.

The usefulness of such a general theory was, in effect, called into question nearly thirty years ago by Delassus. The author (in chapter II) examines critically the nature of such objections. The situation is somewhat as follows: nobody has ever encountered in nature a dynamical system with nonlinear constraints (that is, nonlinear in the velocities). Furthermore, all efforts to construct an artificial example of such a system S by isolating it as a partial system from a larger system $S+S'$ is bound to result in failure unless the whole system $S+S'$ likewise has nonlinear constraints. Here S' is conceived as having zero mass and as being acted on by no forces. It is introduced merely as a mechanism for transmitting to S the proper forces of reaction. Nevertheless, the author feels that a general theory of nonlinear constraints is not entirely useless, first because it has not been proved that a natural system with nonlinear constraints will never be encountered, and second (and more especially) because this general theory is analytically equivalent to the problem of determining with the greatest possible accuracy ("avec l'exactitude que permettent les données du problème") the motion of a holonomic system in which the forces are not completely known, though certain first integrals are known.

The last two chapters (V and VI) are, however, independent of this controversy, being restricted to the case of linear (and also homogeneous) nonholonomic constraints. It is also assumed that the kinetic energy is a homogeneous quadratic form in the velocities. In chapter V, the author shows how to generalize (with the help of the symbolic calculus of pseudo-coordinates) and to put into very convenient form the theory of "nonholonomic deviation," due originally to Schieldrop. In chapter VI, the author puts into geometrical form the content of the equations of motion and Hamilton's principle. For this purpose he uses certain Riemann spaces having second order contact with nonholonomic spaces.

D. C. Lewis (Durham, N. H.).

Würschmidt, José. The principles of variation and stationary action. Publ. Inst. Mat. Univ. Nac. Litoral 5, 73–89 (1945). (Spanish) [MF 13723]

Having in mind application to quantum mechanics and with a view to bringing some order into a subject which, according to the author, suffers from lack of uniformity of notations, he gives a comparative study of the principles mentioned in the title. He takes as his starting point the principle of d'Alembert, which he rewrites in a form showing the connections with the above principles and with the functions that occur in their expressions. The discussion in most of the paper is conducted in such a manner that the results are valid whether mass is constant or not; these two

cases, which correspond, respectively, to classical and general (relativistic) mechanics, are separated in the last section, in which a single particle is considered. *G. Y. Rainich.*

Astronomy

Mineur, Henri. Étude théorique du mouvement séculaire de l'axe terrestre. C. R. Acad. Sci. Paris **220**, 107–109 (1945). [MF 13487]

The author sets up a set of differential equations for the motion of the terrestrial axis. He studies the two cases of (a) resonance and (b) nonresonance, neglecting powers of the masses M_i greater than one. In case (a) he uses elliptic integrals to effect a solution of his differential equations. His conclusion is that, if certain initial conditions are imposed, the obliquity of the ecliptic cannot vary more than 10° or 12°; he does not state these initial conditions.

H. E. Buchanan (New Orleans, La.).

Mineur, Henri. Étude du mouvement séculaire du pôle terrestre dans le cas de résonance. C. R. Acad. Sci. Paris **221**, 462–463 (1945). [MF 14687]

This paper is based on the paper reviewed above. The author introduces the initial condition $V=\omega_1 \cos \epsilon_0$, where V has the same period as the secular perturbations of the ecliptic. His solution of the differential equations which define the movement of the pole is then found in terms of elliptic functions. The conclusion is that the period of the oscillation is greater than in the case of nonresonance.

H. E. Buchanan (New Orleans, La.).

Stumpff, K. Über die Reihenentwicklung der rechtwinkligen Bahnkoordinaten im Zweikörperproblem. Astr. Nachr. **274**, 49–68 (1943). [MF 13409]

The author investigates the coefficients of the Taylor series in powers of the time τ of the coordinates in elliptic motion. General expressions are developed for these coefficients, for which only recursion formulae were formerly available. The developments in power series in three variables are finally reduced to power series in two variables by complete summation with respect to τ. The resulting series are satisfactorily convergent for orbits of small eccentricity. The investigation is related to an earlier article by the author [Astr. Nachr. **273**, 105–112 (1942); these Rev. **6**, 189]. *D. Brouwer* (New Haven, Conn.).

Sémirot, Pierre. Conditions de choc dans le problème des trois corps. C. R. Acad. Sci. Paris **218**, 389–391 (1944). [MF 12112]

The author outlines briefly a method of determining conditions satisfied by the coordinates of three bodies in the neighborhood of collisions of various types. The method uses expansions in series of powers of $t^r \log t$, where t is the time variable and r is a suitably chosen function. [The reviewer found the discussion of these series obscure.]

W. Kaplan (Ann Arbor, Mich.).

Sokoloff, G. Sur un nouveau cas d'intégrabilité dans le problème rectiligne des trois corps. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 95–98 (1945). [MF 12704]

The author considers the planar problem of three bodies subject to attractive or repulsive forces depending on the mutual distances. By introduction of suitable coordinates for relative motion, he derives generalizations of the classical triangular and collinear solutions. In addition, he finds a new family of collinear solutions for a force proportional to the third power of the distance. *W. Kaplan.*

Lahaye, Edmond. Le choc quadruple dans le problème des n corps. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 576–590 (1943). [MF 13855]

The author assumes a quadruple collision in the n-body problem, such that the coordinates of three of the bodies relative to a fourth body approach 0 as the time t approaches a definite value t_0. He assumes expansions of the coordinates in fractional powers of $t-t_0$ and determines initial terms of these expansions. *W. Kaplan* (Ann Arbor, Mich.).

Kopal, Zdeněk. Theoretical velocity curves of close binary systems. Proc. Amer. Philos. Soc. **89**, 517–530 (1945). [MF 13527]

The purpose of this investigation is to derive expressions for the contributions to the radial velocities of the components of close binary systems, not due to orbital motion. In order to obtain orbital elements of close spectroscopic binary systems free from systematic effects the observed radial velocities should be corrected for such nonorbital contributions. The distortion of the components due to axial rotation and mutual tidal action and the varying brightness over the surface of the components are introduced. After treating noneclipsing systems the author deals with the important additional effect of mutual eclipses. In a final section the influence of tidal distortion on the elements of spectroscopic orbits is examined. *D. Brouwer.*

Mineur, Henri. Sur la détermination de l'apex au moyen des mouvements propres. C. R. Acad. Sci. Paris **220**, 80–81 (1945). [MF 13482]

This paper presents a form of solution of the problem of finding the solar motion from the proper motions of stars. The dispersion due to errors of observation and that due to the residual velocities of the stars of a group are introduced simultaneously in assigning weights to the observation equations. In this respect the procedure is more general than the more common forms of solution. [The superiority of the method in specific applications would seem to depend largely upon the distribution of the distances of the members of the group of stars treated. This question is not discussed by the author.] *D. Brouwer* (New Haven, Conn.).

Harrison, Marjorie Hall. A stellar model with a gravitational source of energy. Astrophys. J. **102**, 216–222 (1945). [MF 13623]

The equations of stellar structure are integrated numerically, assuming Kramers' opacity law, negligible radiation pressure, and energy-generation proportional to the temperature. The mass and radius of the convective core are found to be, respectively, 0.105 and 0.14 times those of the star. The expressions found for the luminosity and central density, pressure and temperature in terms of mass and radius differ little from those for the point-convective model.

T. G. Cowling (Bangor).

Sen, N. R., and Burman, U. R. On the internal constitution of stars of small masses according to Bethe's law of energy generation. Astrophys. J. **102**, 208–215 (1945). [MF 13622]

It is assumed that stars of nearly solar mass are constructed according to the Cowling model, with convective

cores in which energy is generated according to Bethe's law. It is found that central temperature, hydrogen content, radius and central density can be inferred from mass and luminosity. To central temperatures from 19 to 22 million degrees and hydrogen contents from 0.15 to 0.95 there corresponds a two-dimensional region in the mass-luminosity plane, near the mass and luminosity of the sun. Ten known stars are considered within this region. The observed radii and those predicted from the assumptions are found to be in reasonable agreement. If the central temperature is less than 22 million degrees, stars of more than 3 solar masses and more than 33 solar luminosities are inconsistent with Bethe's law. The authors consider it unlikely that very large stellar masses can be permitted by Bethe's law, even with higher central temperatures. *T. E. Sterne.*

Ledoux, P. On the radial pulsation of gaseous stars. Astrophys. J. **102**, 143–153 (1945). [MF 13621]

For a gravitating mass of perfect gas, $\frac{1}{2}d^2I/dt^2=2T+\Omega$, where I is the moment of inertia with respect to the center, T the translatory kinetic energy of the molecules, Ω the gravitational potential energy. This theorem is applied to find approximately the principal period of radial pulsation of a star; the result agrees with one found earlier, using Rayleigh's principle [Ledoux and Pekeris, same J. **94**, 124–135 (1941)]. The theorem is then put in a form suitable for considering similar pulsations of a rotating star. As an approximation, the pulsation is assumed such that at any instant the radial displacement is proportional to the distance from the center, and the angular velocity is the same at all points of the star. The period is found to be

$$2\pi I^{\frac{1}{2}}\{-(3\Gamma-4)\Omega+(5-3\Gamma)\omega M\}^{-\frac{1}{2}},$$

where ω, M are the angular velocity and angular momentum and Γ is the adiabatic exponent. A similar, but much cruder, formula is found by a different method, using a theorem of Poincaré. *T. G. Cowling* (Bangor).

Dauvillier, M. A. Recherches sur la genèse, la nature et l'évolution des planètes. Arch. Sci. Phys. Nat., Geneva **24**, 5–24, 65–95, 125–159, 272–273 (1942). [MF 14192]

The author presents a hypothesis as to the origin of the solar system: that, due to the perturbing influence of a passing star, two streams of ionized matter were drawn out of the sun's photosphere; these streams spread out as logarithmic spirals under the influence of electromagnetic forces and later condensed to form the planets. The ramifications of the theory are explored in some detail: density and chemical constitution of the planets, formation of satellites, and peculiarities of the internal and surface structure of earth and moon. *W. Kaplan* (Ann Arbor, Mich.).

Esclangon, Ernest. Sur l'influence de l'expansion de l'univers et l'absorption internébulaire dans le dénombrement, par magnitudes, des nébuleuses spirales. C. R. Acad. Sci. Paris **218**, 19–22 (1944). [MF 13444]

It is assumed that all the spiral nebulae are of equal absolute magnitude and that the red-shift is the result of a dissipation of energy in space represented by a factor e^{-2pr} at distance r. The diminution of intensity in the light received is assumed to be $(1+v/c)^{-2p}$, where v, the recession velocity, is defined as dr/dt, with $r=r_0e^{at}$ for the distance at time t. The predicted value of $d\log N/dm$, where N is the number of nebulae counted down to apparent magnitude m, is compared with Hubble's result $\log N=0.50m-2.77$ $(18\leqq m\leqq 21)$ and gives $p<3.5$. On the other hand, if p is taken to be zero there must be a maximum of β equal to 1/525. If $\alpha=0$, so that there is no velocity of recession in terms of the distance variable employed, then the value of β must be 1/800. *G. C. McVittie* (London).

Hydrodynamics, Aerodynamics, Acoustics

Marden, Morris. Axisymmetric harmonic vectors. Amer. J. Math. **67**, 109–122 (1945). [MF 11927]

The author uses S. Bergman's generalization of Whittaker's formula for an investigation of the velocity potential $\Phi(x,\rho)$ and stream function $\Psi(x,\rho)$, $\rho^2=y^2+z^2$, of an axially symmetric flow. Denoting by $f(u)$ an analytic function, the potential Φ is given by a Laplace integral while

$$\Psi=-\rho i(2\pi)^{-1}\int_0^{2\pi}f(x+i\rho\cos t)\cos t\,dt.$$

The author mentions that this formula appears also in a paper by L. Bers and A. Gelbart [Quart. Appl. Math. **1**, 168–188 (1943); these Rev. **5**, 25]. [See also H. Bateman and S. O. Rice, Amer. J. Math. **60**, 297–308 (1938).] The author considers several flows in suitably restricted domains of the meridian plane in their relation to a given $f(u)$. *A. Weinstein* (Toronto, Ont.).

Lin, C. C. On the stability of two-dimensional parallel flows. I. General theory. Quart. Appl. Math. **3**, 117–142 (1945). [MF 12649]

This is the first in a series of three papers. In the introduction the general state of the problem is presented and the results are listed. (i) All symmetrical and all boundary layer velocity profiles become unstable at a finite Reynolds number. (ii) An approximate method of computing the stability limit for a given profile is given. (iii) A physical interpretation of the instability of profiles with an inflection point is presented. The effect of viscosity and the conception of the "critical layer" are discussed in terms of the diffusion of vorticity, and the place of the inflection point profiles in the general theory is clarified.

After a discussion of previous work on laminar stability, the general problem is formulated. It is pointed out that, besides the Couette and Poiseuille flows which are rigorously plane and parallel, the analysis can deal with flow patterns which are only approximately parallel like all problems of the boundary layer type such as the Blasius boundary layer, the plane jet, etc. If $w(y)$ denotes the given velocity profile, $\psi(x,y,t)=\phi(y)e^{i\alpha(x-ct)}$ the stream function of a small perturbation of wave length $\lambda=2\pi/\alpha$ and $c=c_r+ic_i$, the mathematical problem involves the solution of the classical Sommerfeld equation

$$(1)\quad (w-c)(\phi''-\alpha^2\phi)-w''\phi=-(i/\alpha R)(\phi^{iv}-2\alpha^2\phi''+\alpha^4\phi),$$

where R denotes the Reynolds number. The stability problem then becomes an eigenvalue problem; c_i is to be determined as a function of α, R and its sign decides whether the motion is stable or unstable.

Heisenberg and Tollmien found unstable solutions for Poiseuille and Blasius flows, respectively, but their results lacked mathematical rigor. The author presents two methods of solving equation (1). (i) Putting $y-y_0=\epsilon\eta$, $\phi(y)=\chi(\eta)$, where ϵ is a small parameter, ϕ is found from a series

$$\phi(y)=\chi(\eta)=\chi_{(0)}+\epsilon\chi_{(1)}+\cdots;$$

specifically, if y_0 is taken as the ordinate of the "critical layer," where $w=c_r$, $\epsilon=(\alpha R)^{-1}$, the equations for $\chi_{(j)}$ become

$$w_0'\eta\chi_{(0)}''+i\chi_{(0)}^{IV}=0, \qquad w_0'\eta\chi_{(j)}''+i\chi_{(j)}^{IV}=L_{j-1}(\chi), \quad j\geqq 1.$$

The first equation can be solved in terms of Bessel functions of order $\frac{1}{3}$ and since the homogeneous part of all equations is the same the solutions for all $\chi_{(j)}$ can be obtained.

(ii) Asymptotic solutions can be obtained by putting

$$\phi(y)=\phi^{(0)}+(\alpha R)^{-1}\phi^{(1)}+(\alpha R)^{-2}\phi^{(2)}+\cdots.$$

Since, however, $(\alpha R)^{-1}$ occurs as a factor of the term of highest order in equation (1), only two solutions are obtained in this way. Furthermore, the reduced equation which $\phi^{(0)}$ satisfies has a singularity at $y=y_0$ where $w=c$. This singularity has caused much confusion in previous work and is consequently discussed here in much detail.

By putting $\phi=\exp\{\int g dy\}$, equation (1) can be transformed into a nonlinear equation for the function g. Two solutions are then found by setting

$$g(y)=(\alpha R)^{\frac{1}{3}}g_0+g_1+(\alpha R)^{-\frac{1}{3}}g_2+\cdots.$$

The correlation of these asymptotic solutions with the four solutions obtained before and their behavior near the artificial singularity is then discussed. It is found that the asymptotic solutions hold for $-7\pi/6<\arg(\alpha\eta)<\pi/6$. The path near the singularity and the physical significance of the singularity are discussed in detail. Finally, the boundary value problem is considered for specific cases like boundary layer flow, channel flow, etc.

H. W. Liepmann (Pasadena, Calif.).

Lin, C. C. On the stability of two-dimensional parallel flows. II. Stability in an inviscid fluid. Quart. Appl. Math. **3**, 218–234 (1945). [MF 13534]

[Cf. the preceding review.] The earlier results of Rayleigh and Tollmien regarding instability in an inviscid fluid are critically discussed. It is pointed out that the existence of a flex in the velocity profile, on which much emphasis has been placed, is not a decisive factor for the laminar instability problem. In the case of velocity profiles having a point of inflection at $y=y_s$ a neutral disturbance with a phase velocity c equal to the mean velocity $w(y_s)$ at the inflection point always exists. Unstable disturbances in the neighborhood of the neutral mode also always exist. For this theorem, due to Tollmien, a second rigorous proof is given and the condition that $w'''(y_s)\neq 0$ is found unnecessary. It is again emphasized that results regarding the instability in an inviscid flow have to be considered as limiting cases for $R_s\to\infty$ only and care has to be taken in carrying the results over to cases of finite Reynolds number.

The second part of the paper discusses the physical interpretation of the instability of velocity profiles with inflection points, along lines first suggested by von Kármán. Consider the exchange of two fluid particles in the flow. Since the particles come from different regions of the mean velocity profile, they will in general have different vorticity. The interaction of the excess vorticity with the mean vorticity results in an acceleration of the particles. This acceleration is computed in two different ways. It is shown that if the mean vorticity has no extremum there is always a tendency to return a displaced particle. The motion is stable. If the vorticity has an extremum, instability is possible. This consideration gives the physical interpretation for the importance of a flex as a point of maximum vorticity for the stability problem. *H. W. Liepmann.*

Kravtchenko, Julien. Sur une extension des régimes à la Poiseuille. C. R. Acad. Sci. Paris **220**, 647–648 (1945). [MF 14062]

The properties of plane Poiseuille flows are well known. The author studies a cylindrical space flow for which the nonvanishing x-component of the velocity is a function of y and z. For this case, the solution of the Navier-Stokes equations is shown to depend upon the solution of Poisson's equation. As is well known, the solution of Poisson's equation can be expressed as the sum of a particular integral and a harmonic function. For the problem under discussion, the determination of the proper harmonic function leads to a Dirichlet problem. In the case where the (y, z) domain is bounded by a rectangle, the author states that the solution of the Dirichlet problem can be expressed in terms of elliptic functions. *N. Coburn* (Austin, Tex.).

Vazsonyi, Andrew. On rotational gas flows. Quart. Appl. Math. **3**, 29–37 (1945). [MF 12083]

General properties of a compressible viscous flow are studied. The effects of the second viscosity coefficient and the lag of vibrational energy of the molecules behind its equilibrium value are, however, not considered. The general theorems are then applied to frictionless but rotational flows. The theorems of Kelvin, Helmholtz, Bjerknes and Crocco are obtained as special cases. *H. S. Tsien.*

Daymond, S. D. Some solutions of the hodograph equation governing the two-dimensional flow of a compressible fluid. Quart. J. Math., Oxford Ser. **16**, 78–85 (1945). [MF 14168]

This is the same problem of compressible flow rounding a corner, treated by the same method, as in papers by F. Ringleb [Z. Angew. Math. Mech. **20**, 185–198 (1940); these Rev. **2**, 169] and by H. Kraft and C. G. Dibble [J. Aeronaut. Sci. **11**, 283–298 (1944)]. The second particular example in the paper is the same as one of Ringleb's. The statement by the author that the physically possible flow in this case is purely subsonic is, however, erroneous.

H. S. Tsien (Pasadena, Calif.).

Emmons, Howard W. The numerical solution of compressible fluid flow problems. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **932**, 32 pp. (21 plates) (1944). [MF 14539]

Numerical methods have been developed for obtaining the steady, adiabatic flow field of a frictionless, perfect gas about arbitrary two-dimensional bodies. The solutions include the subsonic velocity regions, the supersonic velocity regions, and the transition compression shocks, if required. Furthermore, the rotational motion and entropy changes following shocks are taken into account. Extensive use is made of the relaxation method.

From the author's summary.

Mercier, André. Sur le champ du tourbillon dans les fluides. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 239–243 (1942). [MF 14211]

Bogolyubov, N. N. Statistical theory of turbulence. Uchenye Zapiski Moskov. Gos. Univ. Fizika **77**, 74–100 (1945). (Russian) [MF 15110]

Roy, S. K. Equilibrium of a local vortex. Proc. Benares Math. Soc. (N.S.) **5**, 45–67 (1943). [MF 14443]

This paper is a continuation of the author's study on two-dimensional motion of a vortex around a corner [same

Proc. (N.S.) **3**, 71–93 (1941); these Rev. **5**, 134]. The case of a straight projection from the vertex into the flow on the bisecting line of the corner is treated in detail with the simple main flow along the walls of the corner. Equilibrium positions of the vortex are calculated and the results are given in tables and graphs. *H. S. Tsien.*

Legras, Jean. Généralisation de la théorie du segment portant au cas de l'aile en dérive. C. R. Acad. Sci. Paris **221**, 228–229 (1945). [MF 14255]

The author points out the error in the paper of Theilheimer [J. Aeronaut. Sci. **10**, 101–104 (1943); these Rev. **4**, 228] which has already been discussed by Gail [J. Aeronaut. Sci. **10**, 320 (1943)] and Lin [J. Aeronaut. Sci. **11**, 195–196 (1944)]. To avoid the difficulty of infinite induced velocities at the yawed lifting line he proposes to determine the effective induced angle of attack by use of Munk's integral for the circulation, using the following approximate expression for the induced velocity:

$$w=-2\sin\alpha\log|u|\Gamma'(y)+w_1(u,y),$$

where α is the angle of yaw of the wing, $\Gamma(y)$ is the circulation function, u is the chordwise coordinate, and $w_1(u, y)$ is finite as $u\to 0$. The result shown indicates that the Munk integral is then integrable and yields a correction to the ordinary Prandtl theory. The details are not given and the reviewer was unable to verify the result. *W. R. Sears.*

Legras, Jean. Contribution à la théorie de la surface portante. C. R. Acad. Sci. Paris **220**, 298–300 (1945). [MF 13944]

The author shows that, in the theory of thin lifting surfaces, the disturbance potential $\Phi(x, y, z)$ in the total potential $U\{x+\Phi(x, y, z)\}$, U being the free stream velocity in the x-direction, can be decomposed as $\Phi=\Phi_0+\Phi_1$, where Φ_0 is an irrotational flow without lift and Φ_1 is the part contributing the lift and thus gives the trailing vortex sheet.

H. S. Tsien (Pasadena, Calif.).

Jones, Robert T. Correction of the lifting-line theory for the effect of the chord. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **817**, 7 pp. (2 plates) (1941). [MF 14397]

The usual Prandtl-Lanchester wing theory of large aspect ratio is based on two-dimensional flow around an airfoil with a correction for the downwash due to a trailing vortex sheet calculated on the simplified concept of the lifting line. For small aspect ratios, the concept of the lifting line is not a good approximation. The author proposes to remedy this by considering the ratio of velocities at the trailing edge for flows without circulation around an elliptic disc and an infinitely long strip. Since the circulation around the airfoil is proportional to the velocity at the trailing edge according to the Kutta-Joukowsky condition, the lift calculated by the Prandtl-Lanchester theory should be reduced by the same ratio. The ratio E is found to be equal to the ratio of the semiperimeter and the span. For an elliptical wing, then, the lift coefficient C_L is given as a function of angle of attack α and aspect ratio A by $C_L=2\pi\alpha A/(EA+2)$. This rough correction factor is found to give results which are in good agreement with lifting surface theories, even for wings of rectangular planform. *H. S. Tsien.*

Hildebrand, Francis B., and Reissner, Eric. The influence of the aerodynamic span effect on the magnitude of the torsional-divergence velocity and on the shape of the corresponding deflection mode. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **926**, 45 pp. (4 plates) (1944). [MF 14404]

If $I(y)$ and $e(y)$ are given functions describing certain elastic properties of a wing as functions of the spanwise coordinate y, and if $F(y)$ describes the lift distribution, $\alpha(y)$ the undeformed twist, $c(y)$ the chord, and $\vartheta(y)$ the elastic twist due to aerodynamic loading, the problem of the elastic wing can be stated as follows:

$$\{I(y)\vartheta'(y)\}'+\beta^2 e(y)F(y)=0, \tag{1}$$

$$F(y)/c(y)+(\mu/\pi)\int_{-1}^{1}\frac{F'(\eta)d\eta}{y-\eta}=\alpha(y)+\vartheta(y). \tag{2}$$

Here μ is a given (positive) parameter, while β^2 is another parameter proportional to the dynamic pressure of flight. The integral is a Cauchy principal value. The torsional-divergence speed is said to occur when β^2 is such that the above equations have a nontrivial solution with $\alpha(y)\equiv 0$, that is, when an initially untwisted airfoil may become twisted.

The authors begin by neglecting the integral in (2); this is the common theory in which induction effects are disregarded. The problem can then be solved explicitly with certain simple forms for $c(y)$, $e(y)$, and $I(y)$, and can be solved in general by successive approximations. Proceeding to the solution of the complete system (1), (2), they again employ a method of successive approximations. They propose as the initial assumption for $\vartheta(y)$ the function $\vartheta_1(y)$ giving the elastic twist due to uniform lift distribution; the subsequent procedure is fairly obvious. It is stated that in many practical cases a second approximation to $\vartheta(y)$ is not necessary to evaluate β satisfactorily. Several numerical examples are carried out, especially to compare this theory with the cruder theory mentioned above. It is found that the torsional-divergence velocity may be underestimated as much as 40 percent if finite-span effects are disregarded.

W. R. Sears (Inglewood, Calif.).

Reissner, Eric. On the general theory of thin airfoils for nonuniform motion. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **946**, 63 pp. (1 plate) (1944). [MF 14403]

The differential equations of motion and the boundary conditions for the subsonic nonuniform motion of a wing of any planform are first set up with considerable generality, under the limitations of the "thin airfoil theory," that is, small camber and incidence, velocities small compared to the velocity of flight, and finite velocities at the trailing edge. The problem is then specialized for the case of simple harmonic motion and for incompressible flow, and is expressed by an integral equation. This integral equation is obtained without recourse to vortex motion but can also be derived by application of the Biot-Savart rule. Next, the integral equation is specialized for the airfoil of rectangular planform. As examples, the special cases of the two-dimensional airfoil in nonuniform motion and the finite wing in steady motion are considered. In these cases no new results are obtained; in the latter case the reduction to the ordinary Prandtl lifting-line equation is shown.

For the finite wing in nonsteady motion, approximations are made in certain integrals to be carried out over the region of the wing itself; these are analogous to the approximations that lead to the Prandtl theory for steady motion.

The result is an integral equation having the same terms as the corresponding two-dimensional result and in addition a finite-span correction term, which is similar to that of the Prandtl theory but varies in a simple manner across the chord. A known result of potential theory is used to invert this equation, yielding an integral equation for the tangential velocity at the airfoil and, finally, for the circulation. Formulas for calculation of section lift and moment are given.

The last-mentioned integral equation is compared with an analogous result of Cicala [L'Aerotecnica **18**, 412–433 (1938)=Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **887** (1939)]. Criticism is also made of the theories of Küssner [Luftfahrtforschung **17**, 370–378 (1940)=Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **979** (1941); these Rev. **2**, 331] and Sears [Proc. Fifth International Congress for Applied Mechanics, 1938, pp. 483–487]. *W. R. Sears.*

Silber, Robert. Sur la limitation de la quantité de mouvement et son effet sur la sustentation des ailes d'avion. C. R. Acad. Sci. Paris **221**, 86–88 (1945). [MF 14241]

In a previous paper [same C. R. **220**, 162–164 (1945); these Rev. **7**, 93] the author proposed to substitute rot $\rho\mathbf{V}=0$ for rot $\mathbf{V}=0$ for flows of compressible fluids, ρ being the density and $\mathbf{V}$ the velocity vector. Then the methods of solving incompressible flow problems can be completely taken over with the only change of substituting the momentum ρV for the velocity V. If the subscript ω denotes quantities referred to the stagnation condition and c the velocity of sound, it is known that $\sigma=\rho V/\rho_\omega c_\omega$ has a maximum at $V=c$, corresponding to the throat of a De Laval nozzle. Let the subscript 0 denote free stream conditions and let M be the Mach number. Then $\sigma_0=M_0/(1+\frac{1}{2}(\gamma-1)M_0^2)^{\frac{1}{2}+1/(\gamma-1)}$, where γ is the ratio of the specific heats. The maximum value of σ is

$$\sigma_{\max}=\{\tfrac{1}{2}(\gamma+1)\}^{-\frac{1}{2}-1/(\gamma-1)}.$$

For the flow over an airfoil, a complex potential $f(z)$ can be found in the "circle plane" Z together with the transformation from Z to z such that

$$f(z)=\sigma_0\{(Z+a^2/Z)-2ia\sin\alpha\log Z\},$$

$$z=Z+a^2/Z+\sum_1^p a_i/Z^i,\quad f'(z)=\sigma e^{-i\theta},$$

where α is the angle of attack and θ is the inclination of the velocity vector. By increasing the free stream Mach number M_0, the value of σ over the surface of the airfoil will also increase. But if at a critical value of σ_0, say σ_{cr}, σ reaches $\sigma_{\max}$, no further increase is allowed. The author then asserts that the distribution of σ over the airfoil surface is frozen at the pattern which corresponds to σ_{cr}. For $\sigma_0>\sigma_{cr}$, a supersonic region is said to develop with accompanying shock. This concept of the behavior of flow is used to explain the decrease of lift coefficient at high subsonic free stream Mach numbers.

However, as pointed out in the review of the previous paper, the validity of the fundamental hypothesis rot $\rho\mathbf{V}=0$ is seriously in doubt. The present concept of the compressibility effect, although it yields results indicating the trends observed by experiments, cannot be considered as correct. *H. S. Tsien* (Pasadena, Calif.).

Silber, Robert. Sur la répartition des pressions sur une aile. C. R. Acad. Sci. Paris **221**, 226–228 (1945). [MF 14254]

This paper continues the discussion of the paper reviewed above. The author also makes some remarks about flows with supersonic free stream. *H. S. Tsien.*

Siestrunck, Raymond. Les relations de symétrie dans le problème de l'hélice; applications aux répartitions optimum. C. R. Acad. Sci. Paris **219**, 151–153 (1944). [MF 14508]

Risack, M. Note sur le calcul de la résistance opposée au mouvement d'un solide par un fluide naturel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 355–366 (1943). [MF 13849]

The author outlines a method for calculating the resistance of a body in the steady flow of a viscous fluid. The total resistance is subdivided into two parts: R_1 due to the pressure, R_2 due to the shear stresses. The author is mainly concerned with R_1, which is divided into two components: the induced resistance R_i due to the vorticity and the wake resistance R_s. To calculate these resistances, the author introduces the equipotential Bernoulli surfaces (or curves in two dimensional motion) (1) $p+\frac{1}{2}\rho V^2=$constant. Due to viscosity, these surfaces are no longer composed of stream lines and vortex lines. Hence the boundary of the body in two dimensional flow is no longer a curve lying in a surface (1). However, the author assumes that a limiting Bernoulli surface exists. By applying Green's theorem to the region bounded by this limiting surface and the body, he obtains expressions for R_i and R_s. These formulas are evaluated for the two dimensional case and R_i is shown to agree with the value found by Prandtl for a perfect fluid. *N. Coburn.*

Velikanov, M. A. Bed load movement. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 190–192 (1945). [MF 13741]

The paper describes how a theoretical bed-load formula can be based on the statistical concept of bed-load movement and on the findings of turbulence studies. Bed-load particles are assumed to move in single steps as proposed by H. A. Einstein. The duration of the rest periods is expressed by the probability that the horizontal turbulence velocities near the bed will overcome the weight of the particles, while the duration and consequently the distance of travel in each step is expressed by the probability that the particle will settle back into the bed against the vertical turbulence velocities. The final formula for the total solid discharge p in unit time and unit width,

$$p=ADW(\tfrac{1}{2}-\phi(\mu))(\tfrac{1}{2}-\phi(\nu))/(\tfrac{1}{2}+\phi(\nu)),$$

contains a constant A, the diameter D of a grain, its horizontal velocity W during the step, the Gaussian probability density $\phi(x)=(2\pi)^{-\frac{1}{2}}\int_0^x e^{-\frac{1}{2}z^2}dz$, and the parameters $\mu=(c(gD)^{\frac{1}{2}}-\bar{u})/\sigma_x$ and $\nu=w/\sigma_z$, where c is a second constant, g is the acceleration due to gravity, $\bar{u}$ is an average velocity in the direction of the flow near the particle, σ_x and σ_z are the root-mean-square values of the velocity fluctuations near the particle in the horizontal and vertical directions and w is the settling velocity of the particle in still water. No application of the formula to measurements is given nor are values known for the constants or the different velocities involved. Only if a set of universally applicable constants can be found will the approach prove its practical value. *H. A. Einstein* (Pasadena, Calif.).

Brun, Edmond, et Vasseur, Marcel. Dépôt, sur un obstacle, de particules en suspension dans un courant fluide. C. R. Acad. Sci. Paris **219**, 112–114 (1944). [MF 14509]

The general equations of motion for small particles suspended in a liquid [cf. Brun, Fasso and Vasseur, same C. R. **218**, 341–344 (1944); Brun and Vasseur, same C. R. **218**, 636–638 (1944); these Rev. **7**, 140, 141] are used in calcu-

lating the number of fog droplets which hit a cylindrical obstruction. The paths of the droplets are determined by a graphical method. Experimental verification of the results was obtained by measuring the heat necessary to evaporate the droplets which hit the cylinder. *H. A. Einstein.*

Leibenson, L. S. A fundamental law of gas motion through a porous medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 16–18 (1945). [MF 13751]

The author generalizes some results which appeared in his book [Underground Hydraulics of Water, Petroleum and Gas, Moscow, 1934]. By dimensional analysis he obtains a formula for the Reynolds number in the form

$$\sum_n A_n\{(\partial p/\partial x)D^2GV^{-2}\}^n,$$

where A_n is a numerical coefficient, the motion is turbulent, the filtration flow is in the direction of the x-axis, p is the pressure, D is the mean effective diameter of a particle of the filtrating medium, G is the density of the fluid and V is its absolute viscosity. He also outlines a general method of determining a laminar or turbulent motion of a gas or of an incompressible fluid through a porous medium.

I. Opatowski (Chicago, Ill.).

Galin, L. A. Unsteady filtration with a free surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 246–249 (1945). [MF 14024]

The author considers the problem of filtration of a fluid through a porous medium occupying a region between two cylinders. One of the cylinders has a small radius and is the source of the fluid. The boundary B of the other cylinder varies in consequence of the filtration. The pressures on the surfaces of the cylinders are assumed to be known and constant. The direction of the filtration is taken either as horizontal or as vertical. Let z be the complex variable in a cross-sectional plane of the cylinders and $z(Z, t)$ the function mapping the circle $|Z| \leqq 1$ onto the region bounded by B, where t is the time. A nonlinear partial differential equation for the function $z(Z, t)$ is derived. In the case of a horizontal flow, if $z(Z, 0)$ is a polynomial of degree n, then $z(Z, t)$ is also a polynomial in Z of degree at most n. The coefficients of this polynomial are determined by a system of nonlinear differential equations of the first order.

I. Opatowski (Chicago, Ill.).

Čarnyĭ, I. A. On the most advantageous spacing of series of wells in oil fields with a waterpressure regime. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 29–33 (1945). (Russian) [MF 13703]

Starr, Victor P. Water transport of surface waves. J. Meteorol. **2**, 129–131 (1945). [MF 14554]

Osborne, M. F. M., and Hart, S. D. Transmission, reflection, and guiding of an exponential pulse by a steel plate in water. I. Theory. J. Acoust. Soc. Amer. **17**, 1–18 (1945). [MF 12778]

The authors consider the solution of the problem wherein a pulse of exponential wave form impinges on a disc immersed in a fluid. The problem is idealized to a two-dimensional situation and is split into two parts. One of these, essentially a wave guide problem, is treated at length and the results are used to explain the occurrence and some of the properties of the precursor wave which has been observed in underwater explosions. *G. F. Carrier.*

Blokhintzev, D. Sound receiver in motion. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 22–24 (1945). [MF 13752]

The author has previously developed a theory of sound propagation in a moving medium. He treats here the case of a sound receiver moving in a sound field when its velocity is (I) subsonic, (II) supersonic. In (I), assuming a potential flow pattern in the medium (replacing the moving receiver by a moving medium), he shows that for small sound frequencies and diaphragm size a signal is amplified. For (II) there are apparent paradoxes in a shock wave passing the sound front and absence of reflected waves from the shock front. The author sketches an approximate analysis based on the usual conditions involving entropy change at a shock wave interface and the Hugoniot-Rankine relation connected with mass, momentum and energy conservation at a discontinuity. He concludes that there are two significant wave phenomena: an entropy wave making no pressure contribution at the receiver and a sound wave.

D. G. Bourgin (Urbana, Ill.).

Elasticity, Plasticity

Michlin, S. G. The two-dimensional problem of contact of two semi-infinite elastic media. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 179–184 (1945). (Russian. English summary) [MF 14042]

Two different semi-infinite elastic media are in frictionless contact along the x-axis except over the range $|x| < a$, corresponding to a thin slit. Along this range the applied normal stresses on each side of the slit are prescribed, and the components of stresses and displacements along the rays $x > a$ and $x < -a$ are assumed to be continuous. The problem is to determine the distribution of stress in the regions $y > 0$ and $y < 0$ under the hypothesis that the stresses vanish at infinity. The problem is solved in terms of quadratures with the aid of the formula of Muschelišvili and the integral of Schwarz. The solution can be generalized to include the problem involving several slits distributed along the x-axis.

I. S. Sokolnikoff (Madison, Wis.).

Ramberg, Walter, and Osgood, William R. Description of stress-strain curves by three parameters. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **902**, 13 pp. (15 plates) (1943). [MF 14396]

Rivaud, Jacques. Problème linéaire de l'élasticité dans le cas où les forces sont connues à la frontière. C. R. Acad. Sci. Paris **220**, 645–647 (1945). [MF 14061]

The author outlines some results in the linear elasticity problem for space domains when the forces on the bounding surface σ are known. The purpose of the paper is to show that the displacements in the interior of the space domain can be obtained by solving a system of Fredholm integral equations of the second kind where the integrals are taken over the bounding surface σ of the domain. By introducing the potentials of Somigliana and three unknown densities, the author obtains an expression for the displacement vector at any point in the interior of σ. Next, he expresses the boundary conditions by use of a system of Fredholm integral equations involving the three unknown densities. By studying the associated system of integral equations, he finds that necessary and sufficient conditions for the existence of the

three unknown densities are that the equilibrium relations be satisfied. Further, by applying the method of Giraud to the original system of integral equations, it is stated that the three theorems of Fredholm can be proved for this system. *N. Coburn* (Austin, Tex.).

de Beauregard, Olivier Costa. Définition et interprétation d'un nouveau tenseur élastique et d'une nouvelle densité de couple en électromagnétisme des milieux polarisés. C. R. Acad. Sci. Paris **221**, 175–177 (1945). [MF 14249]

Ruḫadze, A. On the problem of the deformation of naturally twisted bars. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 483–492 (1944). (Georgian. Russian summary) [MF 14613]

Let an initially twisted prismatic rod be bent by a transverse force applied at the center of gravity of the end section and directed along one of the principal axes of the section. If the angle of twist α is given by $\alpha=kz$, where z is the coördinate measured along the length of the rod, and k is a small parameter whose second and higher powers can be neglected, then an effective solution of the problem can be given in terms of four plane harmonic functions and one biharmonic function. This problem has been considered previously by A. Lourie and G. Janelidze [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 227–228 (1939); **27**, 436–439 (1940); these Rev. **2**, 176], whose solution calls for heavy computations. The present paper is based on a method of solution proposed by the author in a paper on bending of nearly prismatic rods [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 123–138 (1942); these Rev. **4**, 180; cf. also his paper in the same Bull. **2**, 609–616 (1941); these Rev. **6**, 83]. *I. S. Sokolnikoff* (Madison, Wis.).

Gorgidze, A., and Ruḫadze, A. Bending of a twisted bar by a couple. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 253–262 (1944). (Georgian. Russian summary) [MF 14612]

Let the lateral surface of a prismatic rod be free of external loads and let the stresses acting on the ends of the rod be statically equivalent to a twisting couple with moment along the length of the rod and to a bending couple with moment directed along one of the principal axes of inertia of the end section. The authors make use of the formulation of a nonlinear theory of elasticity proposed by F. D. Murnaghan [Amer. J. Math. **59**, 235–260 (1937)] to determine the displacements and stresses in a rod subjected to simultaneous bending and torsion. The solution is given in terms of the classical torsion function and a certain other function satisfying the equation of Poisson. [Cf. the authors' papers in the same Bull. **2**, 397–404, 491–498 (1941); these Rev. **6**, 82.] *I. S. Sokolnikoff* (Madison, Wis.).

Mikeladze, Š. E. On the problem of longitudinal bending of rectilinear bars within the limits of elasticity. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 175–213 (1943). (Russian. Georgian summary) [MF 11689]

Using results of his previous papers [same Trav. **7**, 47–63 (1940); C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 400–402 (1940); these Rev. **3**, 155; **2**, 198] and employing the method of difference equations the author gives a new procedure for the determination of the smallest critical force P at which a bar of rectangular cross-section begins to buckle under a pressure in the direction of its axis. If we assume that the x-axis coincides with the axis of the bar, and the origin with one end of it, then the equation for the displacement y (in the direction perpendicular to the x-axis) is $EI(x)y''=-P(y-f)-Q(l-x)-M$. Here $I(x)$ is the moment of inertia; M, $Q(l-x)$ and f are quantities determined by the method of fixing (clamping or hinging) the bar. The equation is replaced by a system of equations

$$(\sigma_i+k/12n^2)y_i''-(2\sigma_{i+1}-10k/12n^2)y_{i+1}''+(\sigma_{i+2}+k/12n^2)y_{i+2}''=0,$$

$i=0, 1, \cdots, n$, where $k=Pl^2/EI$, $\sigma_i=I_i/I$ and l is the length of the bar. Using his previous results [same Trav. **9**, 49–60 (1941); these Rev. **4**, 91] the author proves that a certain determinant whose elements depend on k must vanish if the above equations have nonvanishing solutions y_i. Numerical examples for different methods of fixing are computed. If $I(x)$ and its four derivatives are continuous, then, as the author shows, k is independent of the method of fixing the endpoints of the bar. *S. Bergman.*

Ylinen, Arvo. Erweiterung der Bernoullischen Biegungstheorie auf den unelastischen Bereich. Ann. Acad. Sci. Fennicae. Ser. A. **57**, no. 7, 15 pp. (1941). [MF 14652]

The paper is concerned with the bending of a prismatic bar which is stressed by an eccentric axial force P. The bar is assumed to be symmetric with respect to the plane through its axis and the line of action of P. In the neighborhood of the strain ϵ_s at the centroid of the cross section the stress-strain relation of the material is developed into a Taylor series. The condition that the stresses transmitted across a section of the bar must be in equilibrium with the load P then furnishes series for the strain ϵ_s and the angle of flexure per unit length of the bar. [Throughout the paper the term "inelastic" should be replaced by "nonlinearly elastic" because the development of the stress-strain relation into a Taylor series implies that the same law holds for loading and unloading. As can be seen from equation (18) of the paper, the neutral axis changes its position as the load increases. Unloading will therefore occur in the neighborhood of the neutral axis and the author's deductions are incorrect unless the same stress-strain relation governs both loading and unloading, that is, unless the material is (nonlinearly) elastic.] *W. Prager* (Providence, R. I.).

Ylinen, Arvo. Über die Knickbiegefestigkeit eines exzentrisch belasteten geraden Stabes und eines zentrisch belasteten ursprünglich gekrümmten Stabes. Ann. Acad. Sci. Fennicae. Ser. A. **57**, no. 14, 28 pp. (1941). [MF 14653]

The results of the paper reviewed above are applied to the buckling of an eccentrically compressed straight bar and a centrally compressed bar with slight initial curvature. [As was pointed out in the preceding review, the treatment applies in the case of nonlinear elasticity rather than plasticity.] *W. Prager* (Providence, R. I.).

Galin, L. A. Elastico-plastic torsion of prismatic bars with polygonal cross-section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 307–322 (1944). (Russian. English summary) [MF 12230]

The torsion problem for a prismatic bar of an elastoplastic material consists of finding two continuous functions τ_{xz} and τ_{yz} defined in the domain D (cross-section of the bar) which satisfy the following conditions: (i) on the boundary of D, $\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)=0$, n being the direction of the normal to the boundary; (ii) in one part D_p of D (plastic domain) $\tau_{xz}^2+\tau_{xy}^2=k^2$, k being a given constant; (iii) in the

remaining part of D (elastic domain D_e)

$$\tau_{xz}=G\theta(\partial\varphi/\partial x-y),\quad \tau_{yz}=G\theta(\partial\varphi/\partial y+x),$$

where φ is a harmonic function, G the elastic shearing modulus and θ the angle of twist per unit length. The mathematical difficulty of the problem is due to the fact that the boundary between D_p and D_e is not known. Only very few special cases have been solved, though approximate solutions have been obtained by Nadai by means of a membrane analogy and by Southwell with the relaxation method.

In this paper a rigorous treatment is given for a polygonal cross-section. The author introduces two analytic functions w_1 and w_2 defined in the upper half-plane of an auxiliary ζ-plane; $w_1(\zeta)$ maps this half-plane onto the elastic domain D_e, whereas w_2 is defined by the requirement that in the z-plane $w_2=\partial\varphi/\partial x-i\partial\varphi/\partial y$. It is shown that the two functions $W_1=dw_1/d\zeta$ and $W_2=dw_2/d\zeta$ are linear combinations of two linearly independent integrals of a Fuchsian differential equation. Thus the problem is reduced to the determination of a certain number of constant parameters entering in this equation. An explicit solution is given for the case of a cross-section which closely approximates a square.

L. Bers (Syracuse, N. Y.).

Išlinskiĭ, A. Yu. Three-dimensional deformations of incompletely elastic and viscoplastic solids. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1945, 250–260 (1945). (Russian) [MF 13705]

A great variety of stress-strain relations for visco-elastic, elastic-plastic and ideally plastic materials are presented. For one of these materials, an ideally plastic solid, the complete set of equations of motion is established and discussed briefly. *W. Prager* (Providence, R. I.).

Hildebrand, Francis B., and Reissner, Eric. Least-work analysis of the problem of shear lag in box beams. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 893, 81 pp. (45 plates) (1943). [MF 14393]

In an earlier paper [J. Aeronaut. Sci. 8, 284–291 (1941); these Rev. 3, 96] Reissner has given a discussion of the problem of the stress in cover sheets of box beams by assuming a parabolic distribution of bending stress across the sheets and determining the span-wise variation of the shape of these parabolas by the method of least work. This leads to a second order ordinary differential equation for determining a function known as "shear lag" or a measure of the effect of the shearing deformability of the cover sheets. Some forty special cases are investigated, involving cantilever beams, simply supported beams, and statically indeterminate beams. The cantilever box beams may have uniform or nonuniform sections (tapered widths, depths, or both) under concentrated loads, uniform or nonuniform loads, or general loads represented by trigonometric series. All beams are symmetrical about a spanwise vertical plane, and the loading is such as to induce no torsion effects.

It is concluded that the shear deformation of the cover sheets in cantilever beams is responsible for as much as a 30 percent increase in the bending stress over that predicted by elementary theory. It is found that the shear-lag effect depends largely upon two factors $(\frac{1}{2}l/W_r)(G/E)^{\frac{1}{2}}$ and $t\sigma_b$, where l is the span length, W_r is the half width of the beam at its root, G is the shear modulus, E is Young's modulus, t is the cover thickness and σ_b is the bending stress in the cover sheets by elementary theory. Numerous results are exhibited graphically. *D. L. Holl* (Ames, Iowa).

Hildebrand, Francis B. The exact solution of shear-lag problems in flat panels and box beams assumed rigid in the transverse direction. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 894, 65 pp. (25 plates) (1943). [MF 14394]

The problem of shear lag is treated by the basic equations of the theory of plane stress for an orthotropic material which is rigid in the direction transverse to the span. By employing a well-chosen stress function H^* in the independent variables $(x_1=x(G/E)^{\frac{1}{2}}, y)$, the boundary value problem for the stresses in a cover sheet reduces to the determination of H^* from the differential equation

$$\partial^2H^*/\partial x_1^2+\partial^2H^*/\partial y^2=0,$$

with suitable boundary conditions. It is shown that any analytic function of x_1+iy is $F(x_1+iy)=H^*+i(EG)^{\frac{1}{2}}u$, where H^* is the modified stress function, E and G are elastic moduli, and u is the displacement in the spanwise direction x (or x_1) of the sheet. Two main problems are given detailed treatment. The first concerns the stress distribution in flat panels subjected to axial concentrated loads applied to the sheet by means of edge stiffeners. The second deals with the stresses in doubly symmetrical cantilever box beams. The results in several cases of these problems are compared with those obtained by approximation methods, for example, for the cantilever box beam by the Reissner least work method [see the preceding review]. *D. L. Holl.*

Lundquist, Eugene E., and Schwartz, Edward B. A study of general instability of box beams with truss-type ribs. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 866, 26 pp. (10 plates) (1942). [MF 14389]

The design of truss-type ribs for box beams is theoretically treated with regard to the function of the ribs in stabilizing the compression flange.

From the authors' summary.

Tagliacozzo, Carlo. The critical stress in a parabolic arch articulated at the ends. Anais Acad. Brasil. Ci. 17, 195–208 (1945). (Portuguese) [MF 14550]

Truesdell, C. The membrane theory of shells of revolution. Trans. Amer. Math. Soc. 58, 96–166 (1945). [MF 12774]

The linear theory of thin shells with axial symmetry is derived anew in an interesting way starting from the general three dimensional theory of elasticity. The author obtains the "membrane" theory and the "bending" theory simultaneously by introducing stress resultants and at the same time developing all quantities with respect to powers of the thickness of the shell. The membrane theory arises from the terms of lowest order in such a development, the bending theory from the terms of next higher order. The author then initiates an attack on the difficult question of the validity of such a development.

The question of the behavior of the solutions of the differential equations at singularities of the middle surface of the shell (such as the apex of a cone) is discussed at some length with the object of deciding what sort of conditions might be imposed in order to insure uniqueness without at the same time admitting solutions which are unreasonable from the point of view of mechanics. The author's conditions at a singularity are that the equilibrium equations (but not necessarily the compatibility equations) should be satisfied on approaching the singularity for the terms of all orders in the development with respect to the thickness, even though the development itself may diverge. Later on many cases with singularities are classified as solvable or insolv-

able on the basis of this criterion when applied to the first two terms in the development.

The final two-thirds of the paper is devoted to the membrane theory, that is, the theory in which only the average stresses over the thickness are dealt with. The object is to obtain explicit solutions for a wide variety of different meridian curves and for various distributions of the external pressure, in particular, distributions of the pressure which are not symmetrical with respect to the axis of the shell. A number of different modes of support at the edges are considered and various possibilities regarding conditions at a singularity of the shell are dealt with. Very few solutions of the problem for nonsymmetrical loads have been given previously. The author, with the aid of a stress function introduced by himself and P. Nemenyi [Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943); these Rev. **5**, 84] has added many further solutions in the form of Fourier series in the longitude angle with coefficient functions given in closed form and for the most part in terms of tabulated functions. In a good many cases whole families of solutions depending on one or two parameters are obtained, so that a wide class of special cases with regard to shapes of the meridian curve can be handled readily. The author remarks that these solutions could be useful for practical purposes since almost any case likely to arise in practice might be treated with good approximation and with relatively simple calculations by substituting for it an appropriately selected special case. *J. J. Stoker* (New York, N. Y.).

Seth, B. R. Bending of an equilateral plate. Proc. Indian Acad. Sci., Sect. A. **22**, 234–238 (1945). [MF 14450]

This is a rediscovery of the solution in closed form for the equilateral triangular plate, simply supported along the edges and loaded uniformly. The solution was first given by S. Woinowsky-Krieger [see S. Timoshenko, Theory of Plates and Shells, McGraw-Hill, New York, 1940, p. 293]. *E. Reissner* (Cambridge, Mass.).

Callandreau, Édouard. Sur la plaque circulaire encastrée à appui intermédiaire. C. R. Acad. Sci. Paris **221**, 278–280 (1945). [MF 14498]

A circular plate is clamped at its edges and carries a uniformly distributed load on its upper surface. It is supported at an interior point A in such a way that the deflection at A vanishes. By combining the expressions for the deflection of a clamped circular plate under a uniformly distributed load and under a load concentrated at a point, to obtain zero displacement at A, it is found that the reaction of the support at A is equal to one-fourth of the total uniform load carried by the plate. The result is independent of the position of the point of support. *H. W. March.*

Levy, Samuel. Bending of rectangular plates with large deflections. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **846**, 36 pp. (9 plates) (1942). [MF 14392]

The solution of von Kármán's fundamental equations for large deflections of plates is presented for the case of a simply supported rectangular plate under combined edge compression and lateral loading. Numerical solutions are given for square plates and for rectangular plates with a width-span ratio of 3 : 1. *From the author's summary.*

Levy, Samuel. Square plate with clamped edges under normal pressure producing large deflections. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **847**, 25 pp. (5 plates) (1942). [MF 14391]

Levy, Samuel, and Krupen, Philip. Large-deflection theory for end compression of long rectangular plates rigidly clamped along two edges. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **884**, 30 pp. (3 plates) (1943). [MF 14388]

The von Kármán equations for flat plates are solved beyond the buckling load up to edge strains equal to eight times the buckling strain, for the extreme case of rigid clamping along the edges parallel to the load. Deflections, bending stresses and membrane stresses are given as functions of end compressive load. *From the authors' summary.*

Levy, Samuel, Goldenberg, Daniel, and Zibritosky, George. Simply supported long rectangular plate under combined axial load and normal pressure. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **949**, 11 pp. (13 plates) (1944). [MF 14386]

A solution is presented for the load-strain curve of a simply supported rectangular plate having a width-length ratio of 1 : 4 under combined normal pressure and axial load. The calculations are carried to axial loads considerably in excess of those required to buckle the plate. For some combinations of normal pressure and axial load the plate can be in equilibrium in more than one buckle pattern. Under such circumstances it is possible for the plate to be either buckled or unbuckled depending on the previous history of loading. *From the authors' summary.*

Levy, Samuel, Fienup, Kenneth L., and Woolley, Ruth M. Analysis of square shear web above buckling load. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **962**, 33 pp. (11 plates) (1945). [MF 14385]

Dunn, Louis G. On the distribution of stress in a thin plate elastically supported along two edges at loads beyond the stability limit. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **859**, 30 pp. (37 plates) (1942). [MF 14390]

Donnell, L. H. The stability of isotropic or orthotropic cylinders or flat or curved panels, between and across stiffeners, with any edge conditions between hinged and fixed, under any combination of compression and shear. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **918**, 57 pp. (11 plates) (1943). [MF 14387]

Roy, Maurice. Sur les effets possibles de la viscosité d'un solide dans les oscillations rapides de torsion. C. R. Acad. Sci. Paris **218**, 256–259 (1944). [MF 13390]

The writer suggests a viscous damping term $\mu\partial^3\theta/\partial t\partial x^2$ in the torsional vibration equation for a cylindrical rod clamped at one end. [This suggestion is made periodically in the literature.] Assuming a periodic forcing couple at the free end expansible in a finite number of harmonics and freedom from resonance, the author gives an approximate first order expression (in μ) for the energy dissipated. Since the harmonic terms enter this expression with coefficients proportional to the squares of the frequencies, it is concluded that the temperature change may give a gauge of μ. *D. G. Bourgin* (Urbana, Ill.).